FACULTAD DE INGENIERÍA Y CIENCIAS

Departamento de Matemática y Estadística

Results about the fundamental solution of a non-local in time telegraph equation

THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

Doctor en Ciencias Mención Matemática

Francisco Alegría M.

Advisor: Dr. Juan Carlos Pozo V.

May 19, 2022.

This work has been partially supported by

- ANID-CONICYT for national doctoral 21201357.
- FONDECYT grant 11160295.

Abstract

This thesis is devoted to study some probabilistic topics associated to the fundamental solution of a non local in-time version of the so-called "telegraph equation". As an outcome of our work, we obtained one published paper, entitled "Non-local in time telegraph equation and very slowly growing variances" [4], and another submitted paper entitled "Non-local in-time telegraph equation and telegraph processes with random time" [3].

We point out that the fundamental solutions of the non-local in time telegraph equations that we have considered can be interpreted as the probability density function of a non-markovian stochastic process. In such a context, by applying some Tauberian theorems, we study the asymptotic behavior of the variance of this process at large and short times. We construct infinitely many new examples such that the variance of the process growth as slowly as we want, extending some earlier results. We also show that our approach can be adapted to define new integro-differential operators which are interesting in sub-diffusion processes.

The final part of this thesis we further study the features of the process described above. By using the theory of Volterra integral equations, we obtain an explicit formula for its moments, also we proved that the Carleman condition is satisfied, which shows that the distribution of the process is uniquely determined by its moments. We also obtain an explicit formula for the moment generating function associated to this moments. As a by-product, we proved that the distribution of this process coincides with the distribution of a compound process of the form $T(|W(t)|)$ where $T(t)$ is the classical telegraph process, and $|W(t)|$ is a random time whose distribution is related to a nonlocal in-time version of the wave equation. To this end, by means the so-called subordination principle in the sense of Prüss, we construct the probability density function from the distribution of the classic telegraph process. Our results exhibit a strong interplay between this type of processes and subdiffusion theory.

To my beloved son and wife.

Acknowledgements

First of all, I would like to thank my teachers since I have reached this stage thanks to their guidance and academic support. To my friends who supported me at this time, especially Henry, for receiving me every time an exercise gave me problems and sharing his ideas and knowledge with me. I also want to thank my family, Ernesto (tío), Cata (prima), Guadalupe (mami), and Antonio (papi), for their constant support and for having shared every moment of my training by my side. Thanks to Pily, Toño, and Kamal for their constant concern towards my family and me, about all in this study time. A special thanks to my mother (Antonia); this would not have been possible without her support and sacrifice throughout my life; I love you, mother. I would also like to give special thanks to my tutor, Juan Carlos, for his guidance, support, and patience from the first to the last moment of my doctorate and preparation of this thesis; without him, I do not know if I had succeeded, thanks JC. Finally, I want to thank Panchito and Angie, my support, my reason for moving forward, my restraint. The patience, love, and support they gave me at this stage are unmatched. Thank you for your patience in every step and your support whenever a problematic situation arises. Thank you, son, for your joy every day, and thank you, amoro, for all your dedication and grand gestures of love towards me. Words are not enough to describe the love and how deeply grateful I am to you. I love you with all my heart.

Agradecimientos

En primer lugar me gustaría agradecer mis profesores ya que gracias a su orientación y apoyo académico, he podido llegar hasta esta instancia. A mis amigos que me apoyaron en estos momentos, especialmente a Henry, por recibirme cada vez que algún ejercicio me dio problemas y compartir sus ideas y conocimientos conmigo. También quiero agradecer a mi familia, Ernesto (tío), Cata (prima), Guadalupe (mami) y Antonio (papi), por el apoyo constante y haber compartido cada momento de mi formación a mi lado. Agradecer a Pily, Toño, y Kamal por su constante preocupación hacia mi familia y hacia mi, sobre todo en este tiempo de estudios. Un especial agradecimiento a mi madre (Antonia); ya que sin su apoyo y sacrificio a lo largo de mi vida, esto no habría sido posible; te amo madre. También me gustaría dar un especial agradecimiento a mi tutor, Juan Carlos, por su guía, apoyo, y paciencia desde el primer hasta el último momento de mi paso por el doctorado y confección de la presente tesis; sin él, no se si lo hubiese logrado, gracias JC. Por último, quiero dar las gracias a Panchito y Angie, mi apoyo, mi razón de seguir adelante, mi contención. La paciencia, amor, y apoyo que me brindaron en esta etapa no tienen comparación. Gracias por la paciencia en cada paso y el soporte cada vez que una situación complicada se presentó. Gracias, hijo, por tu alegría día a día y gracias, amoro, por toda tu dedicación y grandes gestos de amor hacia mi. No alcanzan las palabras para describir el amor y lo profundamente agradecido que estoy con ustedes. Los amo con todo mi corazón.

Contents

Introduction

The main objective of this thesis is to study some probabilistic features of the fundamental solution of a non-local in time version of the so-called ***telegraph equation***, which has been recently proposed in [94, Section 4]. In such a work, the authors have established that the fundamental solution of this equation coincides with the probability density function of a stochastic process denoted by $X_k(t)$. The interest on this process lies in the fact that it can be seen as a non-Markovian version of the classical telegraph process $X(t)$, $t \geq 0$.

Roughly speaking, the telegraph process is a one-dimensional random motion performed by a particle that starts at the time instant $t = 0$ from the origin $x = 0$ of the real line $\mathbb{R}$ and moves with some finite constant speed ν. The initial direction of the motion (positive or negative) is taken on with equal probabilities 1/2. The motion is driven by a homogeneous Poisson process of positive rate as follows. As a Poisson event occurs, the particle instantaneously takes on the opposite direction and keeps moving with the same speed ν until the next Poisson event occurrence; then, it takes on the opposite direction again independently of its previous motion, and so on. This random motion has first been studied by Goldstein [37] and Kac [54] and was called the *telegraph process* by its connection with the so-called *telegraph equation*. We will explain these concepts, in detail, in the following sections.

Over the years, the telegraph process has demonstrated to have a deep connection with the theory of diffusion processes. Thus, since $X_k(t)$ can be considered a non-Markovian version of $X(t)$, one may wonder if there is some kind of relationship between $X_k(t)$ and diffusion theory. Throughout the thesis, we exhibit that this process is strongly related to the so-called *anomalous diffusion theory*. To this end, we analyze various features of $X_k(t)$. In particular, we analyze the asymptotic behavior of all its moments, and we show that its distribution coincides with the distribution of some process that are of interest to the subdiffusion theory.

In the following paragraphs, we will show some aspects that motivate the study of this type of equations and why they are attractive to the mathematical community.

Why studying non-local in time Partial Differential Equations?

Continuum Local models have described plenty of natural phenomena since they were introduced centuries ago. However, an important series of processes and phenomena follow different rules, and their natural description is better explained by non-local effects.

The previous statement, translated to mathematical language, concerns the local or non-local nature of the involved differential equation in the model. More precisely, a classical Partial Differential Equation (PDE) is a relation between the values of an unknown function and some of its derivatives. In order to check whether one of these equations is satisfied at a point, we only need to know the values of the solution in an arbitrarily small neighborhood of the point, in such a way that all the involved derivatives can be computed at once. This is essentially the local nature of the model. On the contrary, non-local models are characterized by PDEs for which, in order to check whether they are satisfied at a particular point, one needs information about the values of the solution in extensive regions, not small in principle. Most of the times, this is because the equation involves integral operators in its formulation. Such PDEs are known in the literature as non-local partial differential equations (NPDEs).

Non-local models appear in many areas of knowledge, and its theoretical study has undergone a rapid development in recent decades. To a large extent this has been due to its application in problems coming from mathematical-physics and applied sciences, such as viscoelasticity, weak dispersion, anomalous diffusion, economic models involving jumps processes, fluid mechanics, electrodynamics with memory, among others, see, e.g. [15, 27, 31, 36, 68, 69, 71, 75, 83, 96, 105, 110, 114]. In many situations a non-local model gives a significantly better description than a local model. Without being exhaustive, in fluids, the quasigeostrophic equation is a nonlocal model which is used in oceanography to describe the temperature on the surface of the water, see e.g. [28]. In shallow water waves, the Kadomtsev-Petviashvili (KP) equation [55] models long and weakly nonlinear waves propagating essentially along the x direction, with a small dependence in the y variable. In heat diffusion, the Gurtin-Pipkin equation [41] is a non-local model introduced in order to avoid the so-called "*infinite speed of heat propagation paradox*" that the classical heat-equation originates.

From a mathematical point of view, it is worth saying that the behavior and properties of the solutions to a NPDE could be completely different from those of the solutions of a classical PDE, even when they have a similar structure. This occurs for instance in the following equation modeling fractional diffusion

$$D_t^\alpha(u(t,x)-u_0(x))-\Delta u(t,x)=0, \quad x\in\mathbb{R}^d,\ t\geq 0, \tag{1.1}$$

where u_0 plays the role of initial data, $\alpha\in(0,1]$ and the operator D_t^α denotes the time-fractional derivative in the sense of Riemann-Liouville, which, for a regular enough function f, is defined by

$$D_t^\alpha f(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{f(s)}{(t-s)^\alpha}ds, \tag{1.2}$$

where $\Gamma(z)=\int_0^\infty x^{z-1}e^{-x}dx$, $Re(z)>0$, stands for the Gamma function. Indeed, it has been established in [56, Section 3] that if $\alpha\in(0,1)$ then the fundamental solution of (1.1) belongs to $L_p(\mathbb{R}^d)$ if and only if $1\leq p<\frac{d}{d-2}$, provided that $d\geq 3$. Whereas, if $\alpha=1$, then the equation (1.1) coincides with the diffusion equation

$$\partial_t u(t,x)-\Delta u(t,x)=0, \quad x\in\mathbb{R}^d,\ t\geq 0, \tag{1.3}$$

whose fundamental solution is the so-called heat kernel, which belongs to $L_p(\mathbb{R}^d)$ for any $p\geq 1$ without any restriction on the dimension. Consequently, the non-local character of the model in the zone has a tremendous influence on the features of the solutions of the corresponding equation.

On the other hand, the time-fractional derivative in the sense of Riemann-Liouville defined in (1.2) corresponds to one of the most canonical examples of non-local in time operators that appears in the literature. This is mainly because time-fractional derivatives are closely related to a class of Montroll-Weiss continuous-time random walk models, see e.g., [106, 111], and they have became one of the standard physics approaches to model anomalous diffusion processes, see e.g., [62, 63, 104].

Roughly speaking, an anomalous diffusion process is any generalized diffusion process with a non-linear relationship between the mean squared displacement (MSD) and time $t>0$. Depending on such growth rate of the asymptotic behavior, there are several types of anomalous diffusion processes. For example, sub-diffusive processes correspond to those processes where the MSD grows slower than a linear function (at least asymptotically). The most typical behavior of the MSD in subdiffusion processes follows a power-law of the form ct^α, is where c a constant and $\alpha\in(0,1)$. The details of this type of derivation from physics principles and for further applications of such models can be found in [83]. Thenceforth, problems of the form (1.1) (and nonlinear variants) have received a lot of attention, see e.g., [60, 61, 67, 83, 92] and the references therein for the physical background.

Anomalous diffusion processes appear in several fields, for example: daily fluctuations in climate variables as temperature [66], stock price variations [50], worm-like micellar solutions [35], heartbeat intervals and in DNA sequences [102]. They also appear in the theory of heat conduction with memory and diffusion in porous media with memory [19, 30, 41, 49, 76]. A different context where anomalous diffusion processes appear is the theory of stochastic processes. For instance, the fractional Brownian motion [40, 74], semi-Markov processes [88, 101] and stochastic processes with randomly varying times [14, 77, 80, 81, 83, 86, 87, 88, 89] can be viewed as anomalous diffusion processes.

One of the most important mathematical settings that gives rise to anomalous diffusion models is the theory of non-local evolution equations. We refer the reader to [56, 57, 58, 93, 94, 113] for recent works on this topic. We point out that all these works provide a solid background in the study of non-local equations of evolution, which motivate us to make a contribution to the theory of this type of equations.

Now we are going to explain, in more detail, the topic of this thesis introducing the so-called *telegraph equation* and the *telegraph processes*.

1.1 Non-local in time telegraph equation

As we have mentioned before, the one-dimensional telegraph process $\{X(t)\}_{t\geq 0}$ is a stochastic process that describes the motion of a particle that travels at constant speed $\nu > 0$ on the real line, and switches the movement direction at a sequence of random times according to the arrival epochs of a homogeneous Poisson process $\{N(t)\}_{t\geq 0}$ of rate $\eta > 0$. This process was introduced to represent a random motion with finite velocity, in order to overcome the severe limitations of the *Brownian motion process* in the realistic representation of real random motions, such as infinite speed with which it travels the trajectories or the non-differentiability of trajectory (absence of inertia). The telegraph random motion was first studied by Goldstein [37] and Kac [54], and it received that name due to its connection with the telegraph equation. Indeed, it can be proved that the probability density function $P(t, x)$ of $X(t)$ satisfies the *telegraph equation*

$$\partial_t^2 u(t,x) + 2\eta\, \partial_t u(t,x) - \nu^2 \partial_x^2 u(t,x) = 0, \quad t > 0,\, x \in \mathbb{R}, \tag{1.4}$$

subject to the initial conditions

$$u(0,x) = \delta_0(x), \qquad \partial_t u(t,x)\big|_{t=0} = 0,$$

where δ_0 stands for the delta Dirac distribution. In other words, the probability density function of the telegraph process coincides with the so-called ***fundamental solution*** of the equation (1.4). Additionally, Kac [54] established that if $\nu^2/\eta \to 1$ as $\nu, \eta \to \infty$, then the probability density function $P(t, x)$ converges to the heat kernel

$$H(t,x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{|x|^2}{2t}\right), \quad t > 0,\ x \in \mathbb{R}, \tag{1.5}$$

which is the probability density function of the ***standard Brownian motion***. This fact shows that there is a strong connection between the telegraph process and the diffusion theory, which has been exploited in different fields, such as statistical physics, financial modeling, transport phenomena, and hydrology, among others. We refer the reader to [7, 16, 64, 65] for theoretical and applied results.

In an effort to better understand the theory of diffusion processes, over the years, several extensions of the telegraph process have been proposed. For instance, Cascaval, Eckstein, Frota and Goldstein [23] have considered the so-called ***time-fractional telegraph equation***

$$D_t^{2\alpha} u(t,x) + 2\eta\, D_t^{\alpha} u(t,x) - \nu^2 \partial_x^2 u(t,x) = 0, \quad t > 0,\, x \in \mathbb{R}, \tag{1.6}$$

where $\alpha \in (0, 1)$, and the time-fractional derivatives $D_t^{2\alpha}$ and D_t^{α} must be understood in the sense of *Caputo*, which, for $\beta > 0$, is defined by

$$D_t^{\beta} f(t) = \begin{cases} \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^{\infty} (t-s)^{m-\beta-1} f^{(m)}(s)ds, & \text{if } \beta \in (m-1, m), \\ f^{(m)}(t), & \text{if } \beta = m, \end{cases} \tag{1.7}$$

where $m = \lceil \beta \rceil$. We refer the reader to [8, 72] and the monograph [73] for details about fractional calculus. Cascaval *et al.* proved that, for regular enough initial conditions, the solutions of (1.6) can be approximated by the solutions of the so-called ***time-fractional diffusion equation***

$$2\eta D_t^{\alpha} v(t,x) - \nu^2 \partial_x^2 u(t,x) = 0, \quad t > 0,\ x \in \mathbb{R}. \tag{1.8}$$

This is very interesting since time-fractional equations of the form (1.8) (and nonlinear variants) have been successfully applied in the theory of anomalous diffusion. For instance, Zaslavsky [116] has used an evolution equation of the form (1.8) to model Hamiltonian chaos. Metzler and Klafter in [83] also showed that evolution equations of the form (1.8) describe the transport dynamics in complex systems governed by anomalous diffusion and non-exponential relaxation patterns. Later, Meerschaert and Scheffler [79] showed that equation (1.8) governs a time-changed Brownian motion $B(E(t))$ where $B(t)$ is a Brownian motion, and $E(t)$ is an independent inverse stable subordinator of index $\alpha \in (0, 1)$. All these facts show that there is a deep interplay between the time-fractional telegraph equation (1.6) and the anomalous diffusion theory.

A couple of years later, Orsingher and Beghin [86] have proved that the fundamental solution $U_\alpha(t,x)$ of the equation (1.6) coincides with the probability density function of a stochastic process, which is denoted by $X_\alpha(t)$, $t>0$. This process can be considered a non-markovian version of the telegraph process and has several interesting properties endowed by the nonlocal nature of the time-fractional derivatives. For instance, for any $\alpha \in (0,1)$ the variance of $X_\alpha(t)$ is given by

$$\mathrm{Var}[X_\alpha(t)] = 2\nu^2 t^{2\alpha} E_{\alpha,2\alpha+1}(-2\lambda t^\alpha), \quad t \geq 0, \tag{1.9}$$

where $E_{\alpha,2\alpha+1}$ stands for the Mittag-Leffler function (see [38, Appendix E]). Later, Vergara [112] exploiting the representation (1.9) proved that $\mathrm{Var}[X_\alpha(t)]$ behaves asymptotically as follows

$$\mathrm{Var}[X_\alpha(t)] = \frac{2\nu^2}{\eta\Gamma(\alpha+1)} t^\alpha, \text{ as } t \to \infty.$$

Since $\alpha \in (0,1)$, this implies that $\mathrm{Var}[X_\alpha(t)]$ grows asymptotically at infinity like a sublinear function, which provides additional evidence of the connection between the time-fractional telegraph equation (1.6) and anomalous diffusion. Further, as it has been pointed out in [70, Section 4], for any $\alpha \in (0,1)$ the process $X_\alpha(t)$ is an example of the so-called *inverse subordinators*, which are stochastic processes that have been successfully applied in of anomalous diffusion, see e.g., [9, 78, 82].

One of our main purposes is to further analyze the interplay between this type of processes and the anomalous diffusion theory. To this end, we study a family of stochastic processes $X_k(t)$, $t \geq 0$, which includes the processes $X_\alpha(t)$ as a particular case. Specifically, for a given function $k \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ we consider a process $X_k(t)$ whose probability density function $U_k(t,x)$ coincides with the solution of the following non-local in time evolution equation

$$\partial_t^2\big(k * k * \big(u(\cdot,x) - \delta_0(x)\big)\big)(t) + 2\eta\,\partial_t\big(k * (u(\cdot,x) - \delta_0(x)\big)(t) - \nu^2\partial_x^2 u(t,x) = 0, \quad t>0,\ x\in\mathbb{R}, \tag{1.10}$$

where δ_0 stands for the delta Dirac distribution and the symbol "$*$" denotes the convolution on the positive half-line $\mathbb{R}_+ := [0,\infty)$. It is worthwhile to mention that equation (1.10) must be supplemented with the initial condition $\partial_t u(t,x)\big|_{t=0} = 0$, whenever such condition exists. In other words, we consider a process $X_k(t)$ whose distribution coincides with the ***fundamental solution*** of the equation (1.10). This equation has been recently [94, Section 4] proposed as a version of a non-local in time telegraph equation. For more details about this nonlocal in time equation and its physical foundations, see Chapter 3. Precisely in this article, [94], the authors proved that the fundamental solution U_k of (1.10) exhibits interesting properties, one of them being that for a pair of functions $(k,\ell) \in (\mathcal{PC})$, see Definition 2.12 below, U_k can be viewed as the probability density function whose distribution process, denoted by $X_k(t)$, coincides with U_k at time t, recovering the results obtained in [86, 112]. The function ℓ plays a fundamental role in the techniques used to prove this fact, as we will explain later.

The condition $(\mathcal{PC})$ covers several interesting integrodifferential operators with respect to time that appear in the context of subdiffusion equations, cf. [4, 56, 93, 94, 113]. For instance, note that if we consider $k = g_{1-\alpha}$, where g_β is the standard notation for function

$$g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t>0, \quad \beta>0, \tag{1.11}$$

then the equation (1.10) becomes the *time-fractional telegraph equation* (1.6). On the other hand, the authors also give an explicit formula for the variance of the process $X_k(t)$. However, despite having the form of the variance explicitly, knowing more information about the solution is complicated. One of the ways to get this is to know its asymptotic behavior, thus, one of the interesting questions present in the literature is *how slow the behavior of the variance can be?*. Supported by Karamata-Feller Theorem, see Theorem 2.3 below, and consider the kernel $k(t) = \int_0^1 g_\alpha(t)d\alpha$, it can be proved that the variance associated to the fundamental solution of equation (1.10) increases like $c\log(t)$ for some $c>0$ and this is the slowest behavior known so far. Then a natural question arises:

(Q1) *Is there a pair $(k,\ell) \in (\mathcal{PC})$ such that the variance grows slower than a logarithmic function at infinity?.*

In this thesis, we have developed a method to construct infinitely many examples of functions that allows solving affirmatively this and other questions that arise along the way, see [4]. This method was not presented in the literature until now and it has led to the publication of my first research work. It exploits some specific properties of the pairs $(k, \ell) \in (\mathcal{PC})$, and it extends what was known up to this point regarding the asymptotic behavior of the variance of the process $X_k(t)$, and it allows to create new examples of functions whose variance has an asymptotic behavior at infinity, as slow as desired.

On the other hand, in the time-fractional case $X_\alpha(t)$, and for some particular values of $\alpha \in (0,1)$, we know more specific features of the associated process. For instance, Orsingher and Beghin [86, Section 4] established that the probability density function of $X_{\frac{1}{2}}(t)$ coincides with the distribution of the telegraph process $T(t)$ with a Brownian time, that is

$$X_{\frac{1}{2}}(t) \overset{d}{\equiv} T(|B(t)|), \quad t > 0.$$

This means that for $\alpha = \frac{1}{2}$, the fundamental solution of (1.6) can be interpreted as the distribution of stochastic process that describes the motion of a particle moving back and forth on the real line with velocities $\pm\nu$ (switching at Poisson-paced times) for a random time interval of length $|B(t)|$. In other words, the particle is located at time t in the random space interval $(-\nu|B(t)|, \nu|B(t)|)$. This shows that the distribution related to equation

$$\partial_t u(t,x) + 2\eta\, \partial_t^{1/2} u(t,x) - \nu^2\, \partial_x^2 u(t,x) = 0,$$

covers the whole real line and differs substantially from the case of the telegraph process, where the distribution is concentrated on a finite interval (spreading as time elapses) because of the finite velocity of motion.

A few years later, Orsingher and Beghin [87, Section 3] extended the preceding result for $\alpha = \frac{1}{2^n}$ with $n \geq 2$. More precisely, they established that if $\alpha = \frac{1}{2^n}$ then the probability density function of $X_\alpha(t)$ coincides with the distribution of the following process

$$T(|B_1(|B_2(|\cdots(|B_n(t)|)\cdots|)|), \quad t > 0,$$

where T is the telegraph process and $B_1, \cdots, B_n$ are n independent Brownian motions. Moreover, in the limit case $n \to \infty$, this distribution is the characteristic function of a bilateral random variable with density

$$f(x) = \frac{\sqrt{1+2\eta}}{2\nu} e^{-|x|\frac{\sqrt{1+2\eta}}{2\nu}}, \quad x \in \mathbb{R}.$$

To the best of our knowledge, this type of result has only been established for the values described above. Therefore, we want to address the following natural question.

(Q2) *Can a similar result be obtained for any $\alpha \in (0,1)$?.*

In Chapter 5, we show that the answer of the question ***(Q2)*** is affirmative. More precisely, we show that this type of result is valid for any pair $(k, \ell) \in (\mathcal{PC})$ provided that the function ℓ is 1-regular and θ-sectorial with $\theta < \frac{\pi}{2}$ (see Definitions 2.14 and 2.13 below). Indeed, assuming these two conditions, we prove that the distribution of $X_k(t)$ can be constructed via subordination from the distribution of the telegraph process. This can be used to prove that the distribution of $X_k(t)$ coincides with the distribution of a compound process of the form $T(|\mathcal{W}_{2k}(t)|)$ where $T(t)$ is the classic telegraph process, and $\mathcal{W}_{2k}(t)$ is a random time whose distribution is the folded fundamental solution of the time-fractional evolution equation

$$\partial_t^2 (k * k * (u(\cdot,x) - \delta_0(x))(t) = \sigma^2\, \partial_x^2 u(t,x), \quad t > 0,\ x \in \mathbb{R},$$

where σ is a fixed positive constant.

As a by-product, we prove that the fundamental solution $U_k(t,x)$ of (1.10) converges to the fundamental solution $Z_k(t,x)$ of the so-called *generalized sub-diffusion equation*

$$\partial_t (k * (u(\cdot,x) - u_0(x)) - \frac{1}{2}\, \partial_x^2 u(t,x) = 0,$$

as $\eta, \nu \to \infty$ in such a way that $\frac{\nu^2}{\eta} \to 1$, extending the result obtained by Kac [54] to the nonlocal framework.

In the last part of the thesis, we consider a specific pair of functions $(k_\delta, \ell_\delta) \in (\mathcal{PC})$, which is defined via Laplace transform as follows

$$\widehat{k}_\delta(\lambda) = \frac{1}{\lambda}\left(\frac{\lambda - 1}{\log(\lambda)}\right)^\delta \quad \text{and,} \quad \widehat{\ell}_\delta(\lambda) = \left(\frac{\log(\lambda)}{\lambda - 1}\right)^\delta, \quad \lambda > 0,$$

where $\delta \in (0,1)$. In the special case $\delta = \frac{1}{2}$, we prove that the distribution of $X_{k_\delta}(t)$ coincides with the density of the telegraph process with distributed-order Poisson time $\mathcal{P}(t)$, $t > 0$. which is strongly related to the so-called ultra-slow diffusion theory, see e.g. [60, 78]. In addition, we give some results associated with subdiffusion.

1.2 The Moment-Problem and Carleman Condition

In the classical case, it is known that the problem of moments is completely determined by the classical telegraph process. Hence one might wonder if the same is true for $X_k(t)$ or if there is some analytic clog due to its non-local nature. To contextualize this problem, we will first explain some facts about the so-called moment problem.

The term *Moment-Problem* appears for the first time in the work of T. Stieltjes of 1894-1895, *Recherches sur les fractions continues*, see [108, 109], containing a wealth of new ideas, among other, a new concept of integral, the *Stieltjes Integral*. In this paper, he proposes and solves completely the following problem, which he calls: ***Problem of Moments or Moment-Problem***: Find a bounded non-decreasing function $\psi(x)$ in the interval $[0, \infty)$ such that its *moments* $\int_0^\infty x^n d\psi(x)$, $n = 0, 1, 2, ...$, have a prescribed set of values

$$\int_0^\infty x^n d\psi(x) = a_n. \tag{1.12}$$

The problem of moments (1.12), as its generalizations, is an important mathematical problem which has attracted much attention for more than a century. In 1939 Boas, see [12, 13], showed that given an arbitrary sequence $\{a_n\}_{n=0}^\infty$, there is always a function of bounded variation ψ such that (1.12) is satisfied for all $n \in \mathbb{N}$.

Note that if we consider $d\psi(x)$ as a mass distributed over $[x, x + \Delta x]$, then $\int_0^x d\psi(x)$ represents the mass distributed over the segment $[0, x]$. Stieltjes calls $\int_0^\infty x^n d\psi(x)$ the n-th moment, with respect to 0, of the given mass distribution characterized by the function $\psi(x)$. For more details about the Stieltjes Moment-Problem, see [2, 10, 12, 13, 29, 32, 51, 59] and references therein.

An important approach to, and extension of, the work of Stieltjes to the whole real axis $(-\infty, \infty)$ was achieved by H. Hamburger, see [42, 43, 44], namely

$$\int_{-\infty}^\infty x^n d\psi(x) = a_n. \tag{1.13}$$

This extension is by no means trivial. The consideration of negative values of x introduces new factors in the situation. Hamburger makes extensive use of Helly theorem of choice. He fully discusses the convergence in the complex plane of the associated (if it exists) continued fractions. He shows that a necessary and sufficient condition for the existence of a solution of the Moment-Problem (1.13) is the positiveness of all determinants of Δ_n, where

$$\Delta_n = \begin{pmatrix} a_0 & a_1 & \dots & a_n \\ a_1 & a_2 & \dots & a_{n+1} \\ a_2 & a_3 & \dots & a_{n+2} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ a_n & a_{n+1} & \dots & a_{2n} \end{pmatrix},$$

and also gives criteria for the Moment-Problem (1.13) to be determined or indeterminate. R. Nevanlinna, in 1922 exhibits the solution of the Moment-Problem (1.13) and their properties in terms of the functions $I(z;\psi) = \int_{-\infty}^{\infty} \frac{d\psi(y)}{z-y}$, $z \in \mathbb{C}$. To him is due the important notion of *extremal solutions.*

About the same time, M. Riesz (1923) solved the Moment-Problem on the basis of *quasi-orthogonal polynomials*, see [97]. T. Carleman (1923) shows the connection between the Moment-Problem and the theories of *quasi-analytic functions* and *quadratic forms*, see [20]. Hausdorff (1923) gives criteria for the Moment-Problem (1.13) to possess a solution in a finite interval, see [45]. Several works revealed important connections between the Moment-problem and many branches of analysis. Of particular importance are the connections with functional analysis, the theories of functions and the spectral theory of operators.

In 1926, Carleman in [21, Chapter VIII] developed a rather complete treatment of the Moment-Problem and established a condition for (1.13).

Theorem 1.1. (Carleman Condition). *The Moment-Problem* (1.13) *is determinate if*

$$\sum_{n=0}^{\infty} [a_{2n}]^{\frac{-1}{2n}} = \infty,$$

***Remark* 1.1.** *A Moment-Problem* (1.13) *is said to be determinate if it has at most one solution ψ. Otherwise, it is indeterminate. It is worth mentioning that the Carleman condition gave sufficient conditions for determinacy of the Stieltjes* (1.12) *and Hamburger* (1.13) *Moment-Problems.*

We refer the reader to [21, Chapter VIII, Theorem I] and [2, Chapter 1, Section Addenda and Problems] for the proof of this result and several applications.

It is well-known that moments of any stochastic process are one of the most interesting and useful objects both from theoretical and practical points of view. This especially concerns the telegraph process $X(t)$. For example, the knowledge of moments enables to construct various moment-type estimators in statistics, see [48].

Goldstein [37], and Kac [54] stated that the density $U(x,t)$ with $x \in [-\nu t, \nu t]$ and $t \geq 0$, of the distribution $Pr\{X(t) \in dx\}$, satisfies the classical telegraph equation (1.4). In 2012, A. Kolesnik in [64], showed a detailed moment analysis of the Goldstein-Kac Telegraph process, see e.g., [37, 54]. He studied the asymptotic behavior of the moment functions and gave a complete solution of the Moment-Problem (1.13) for the classic telegraph process $X(t)$. He showed that, for an arbitrary $t > 0$, the moment $X(t)$ satisfies the Carleman condition and, therefore, the distribution of $X(t)$ is completely determined by its moments. Also, Kolesnik obtains an explicit formula for the generating function of the moments μ_{2n}, $n \geq 1$. This explicit form of the moment generating function, see Definition 2.26 below, is given by the following Theorem.

Theorem 1.2. *For any $t > 0$ the moment generating function of the process $X(t)$ has the form:*

$$M(z,t) = e^{\frac{-\eta t}{2}} \left[\cosh\left(t\sqrt{\left(\frac{\eta}{2}\right)^2 + \nu^2 z^2} \right) + \frac{\frac{\eta}{2}}{\sqrt{\left(\frac{\eta}{2}\right)^2 + \nu^2 z^2}} \sinh\left(t\sqrt{\left(\frac{\eta}{2}\right)^2 + \nu^2 z^2} \right) \right].$$

The results obtained in Kolesnik, see [64], have served us as main motivation to address the following questions.

(Q3) *Can we obtain a general formula for all the higher-order moments of $X_k(t)$?.*

(Q4) *In the case that such a formula exists, can we use it to obtain more information about $X_k(t)$?.*

To the best of our knowledge, no works have studied representations nor asymptotic behavior of the higher-order moments of the process $X_k(t)$. In this thesis, we give a simple answer to these questions, generalizing the results obtained in [4, 64, 94]. As a by-product, we prove that the moments $M_{2n}(t)$ for an arbitrary fixed $t > 0$ satisfy the *Carleman condition* and, therefore, the *Hamburger Moment-Problem* (1.13) is entirely solved for the process $X_k(t)$. In other words, the distribution of $X_k(t)$ is entirely determined by its moments.

1.3 Summary of main results

In this section, we will mention the main results obtained in this thesis that answer the questions **(Q1)**, **(Q2)**, **(Q3)**, and **(Q4)** posed above. It is worth mentioning that the enumeration of the following theorems, lemmas, corollaries and examples, respect the enumeration of the section and chapter in which they are found. These results are divided into Chapters 4 and 5, each of which corresponds to a paper or a preprint, as follows.

1.3.1 Main results of Chapter 4

This subsection presents the results obtained in the paper *non-local in time telegraph equation and very slowly growing variance*, see [4]. We recall that such a work is related to the following non-local in time telegraph equation, described in (1.10),

$$\partial_t^2\big(k*k*\big(u(\cdot,x)-\delta_0(x)\big)\big)(t)+2\eta\,\partial_t\big(k*(u(\cdot,x)-\delta_0(x)\big)(t)-\nu^2\partial_x^2u(t,x)=0,\quad t>0,\,x\in\mathbb{R},$$

where δ_0 stands for the delta Dirac distribution and the symbol "$*$" denotes the convolution on the positive half-line $\mathbb{R}_+:=[0,\infty)$, and initial condition $\partial_t u(t,x)\big|_{t=0}=0$, whenever such condition exists.

The following result shows the asymptotic behavior of the variance of the stochastic process associated to fundamental solution of (1.10) at large and short times.

Theorem (4.2). *Let $(k,\ell)\in(\mathcal{PC})$, see Definition 2.12 below. If the Laplace transform $\widehat{\ell}$ is a regularly varying function of index $\varrho_1<\frac{1}{2}$, see Definition 2.3 below, then*

$$\mathrm{Var}[X_k(t)]\sim\frac{2\nu^2}{\Gamma(1-2\varrho_1)}\left(\widehat{\ell}(t^{-1})\right)^2,\ \ as\ t\to0^+.$$

Further, if the function $t\mapsto\widehat{\ell}(t^{-1})$ is a regularly varying function of index $\varrho_2>-1$, then

$$\mathrm{Var}[X_k(t)]\sim\frac{\nu^2}{\eta\,\Gamma(1+\varrho_2)}\,\widehat{\ell}\Big(\frac{1}{t}\Big),\quad as\quad t\to\infty.$$

In order to answer the question presented above, *is there a pair $(k,\ell)\in(\mathcal{PC})$ such that the variance grows slower than a logarithmic function at infinity?*, we prove the following Lemma.

Lemma (4.1). *Let $f,g\in L_{1,loc}(\mathbb{R}_+)$. Assume that $f,g\in(\mathcal{CM})$, see Definition 2.2 below, then there exists $h\in(\mathcal{CM})$ such that*

$$\widehat{h}(\lambda)=\widehat{f}(\lambda)\ \widehat{g}\big(\widehat{f}(\lambda)\big),\quad\lambda>0.$$

Consequently, we present the corollary that allows us to construct the examples that answer the question posed in the affirmative. We develop a method to construct new examples such that the variance has a slow growth.

Corollary (4.2). *For all $n\in\mathbb{N}$ there exists a pair $(\phi_n,\psi_n)\in(\mathcal{PC})$ such that $\phi_n\in(\mathcal{CM})$ and*

$$\widehat{\psi_n}(t^{-1})\sim\log^{[n]}(t),\ \ as\ t\to\infty,$$

and

$$\widehat{\psi_n}(t^{-1})\sim t\left(\log(t^{-1})\right)^n\ \ as\ t\to0^+,$$

where $\log^{[n]}=\underbrace{\log\circ\log\circ\cdots\circ\log}_{n-\text{times}}$.

And, with this previous result, we found an example that answers the question affirmatively.

Example (4.7). *Let $n\in\{2,3,\cdots\}$. Consider pair $(k,\ell)=(\phi_n,\psi_n)$ given in Corollary 4.2, then*

$$Var[X_k(t)]\sim\frac{\nu^2}{\eta}\log^{[n]}(t),\quad as\quad t\to\infty,$$

and

$$Var[X_k(t)]\sim\nu^2t^2\left(\log(t^{-1})\right)^{2n},\quad as\quad t\to0^+,$$

where $\log^{[n]}=\underbrace{\log\circ\log\circ\cdots\circ\log}_{n-\text{times}}$.

1.3.2 Main results of Chapter 5

This subsection presents the results obtained in the pre-print entitled *non-local in time telegraph equation and telegraph processes with random time*, [3].

In order to answer the question presented above, *can we obtain a general formula for all the higher-order moments of $X_k(t)$?*, we prove the following Lemma.

Lemma (5.1). *Let $(k,\ell)\in(\mathcal{PC})$, see Definition 2.12 below, and $n\in\mathbb{N}_0$. Then $M_n(t)$ is given by the formula*

$$M_n(t)=\begin{cases}0, & \text{if } n=2m+1, \text{ for some } m\in\mathbb{N}_0,\\ (2m)!\,\nu^{2m}\left(1*\phi_\ell^{*(m)}\right)(t), & \text{if } n=2m, \text{ for some } m\in\mathbb{N},\end{cases}$$

where $\phi_\ell=r_{2\lambda}*\ell$, and $\phi_\ell^{*(m)}$ is defined recursively as follows

$$\phi_\ell^{*(m)}=\begin{cases}\phi_\ell, & m=1,\\ \phi_\ell*\phi_\ell^{*(m-1)}, & m\geq 2.\end{cases}$$

In order to learn more about the distribution and answer the question, *in the case that such a formula exists, can we use it to obtain more information about $X_k(t)$?*, we prove that the moments of the process $X_k(t)$ satisfy the Carleman condition.

Theorem (5.1). *For any fixed $t>0$, the moments of $X_k(t)$ satisfy the Carleman condition*

$$\sum_{m=1}^{\infty}\frac{1}{\sqrt[2m]{M_{2m}(t)}}=\infty.$$

Once the distribution of a stochastic process is proved to be uniquely determined by the moments, one may wonder if a formula of the corresponding moment generating function can be obtained in an explicit form.

Theorem (5.2). *Let $(k,\ell)\in(\mathcal{PC})$, see Definition 2.12 below. For any $t>0$ and $|z|<R_0$, the moment generating function, see Equation (5.11) below, of the process $X_k(t)$ has the form*

$$\boldsymbol{M}(t,z)=\frac{1}{2}\left[\left(\frac{\eta}{\sqrt{\eta^2+z^2\nu^2}}+1\right)s\Big(t,\eta-\sqrt{\eta^2+z^2\nu^2}\Big)+\left(1-\frac{\eta}{\sqrt{\eta^2+z^2\nu^2}}\right)s\Big(t,\eta+\sqrt{\eta^2+z^2\nu^2}\Big)\right],$$

where $s(t,\cdot)$ is the scalar resolvent defined in (2.6) below.

The following theorem extends the results presented by Alegría and Pozo in [4, Theorem 3.2], which are also presented in Chapter 4, Theorem 4.2, and give us a precise description of the asymptotic behavior of the moments when the function ℓ satisfies an additional regularity condition.

Theorem (5.3). *Let $(k,\ell)\in(\mathcal{PC})$, see Definition 2.12 below, and $n\in\mathbb{N}$. If $\widehat{\ell}\in\mathcal{RV}_\infty^{\varrho_1}$ for some $\varrho_1<\frac{1}{2n}$, see Definition 2.3 below, then*

$$M_{2n}(t)\sim\frac{(2n)!\,\nu^{2n}}{\Gamma(1-2n\varrho_1)}\big(\widehat{\ell}(t^{-1})\big)^{2n},\ \text{as } t\to 0^+.$$

Further, if $\ell\notin L_1(\mathbb{R}_+)$ and $\widehat{\ell}\in\mathcal{RV}_0^{\varrho_2}$ for some $\varrho_2>-\frac{1}{n}$, see Definition 2.3, then

$$M_{2n}(t)\sim\frac{(2n)!}{\Gamma(1+n\varrho_2)}\left(\frac{\nu^2}{\eta}\right)^n\big(\widehat{\ell}(t^{-1})\big)^n,\ \text{as } t\to\infty.$$

If $\ell\in L_1(\mathbb{R}_+)$, then

$$M_{2n}(t)\sim(2n)!\,\nu^{2n}\left(\frac{\|\ell\|_1^2}{1+\eta\|\ell\|_1}\right)^n,\ \text{as } t\to\infty.$$

Also, we prove that the fundamental solution of the equation (1.10) can be constructed via subordination from the distribution of the classical telegraph process, answering question **(Q2)**. To this end, we prove the following results.

Proposition (5.3)**.** *Let $\sigma > 0$ and $(k, \ell) \in (\mathcal{PC})$, see Definition 2.12 below. Suppose ℓ is 1-regular and θ-sectorial with $\theta < \frac{\pi}{2}$, see Definitions 2.14 and 2.13 below. Then we have*

$$\partial_\tau(-W_{\ell,\sigma}(t,\tau)) = 2\, V_{2k}(t,\tau), \quad t > 0, \tau > 0,$$

where $W_{\ell,\sigma}(\cdot,\cdot)$ is the propagation associated to ℓ and σ, see Definition 2.16 below, and $V_{2k}(\cdot,\cdot)$ denotes the solution of the following evolution equation

$$\partial_t^2(k * k * (v(\cdot,x) - \delta_0(x))(t) = \sigma^2 \partial_x^2 v(t,x), \quad t > 0,\ x \in \mathbb{R},$$

subject the initial conditions $v(0,x) = \delta_0(x)$ and $\partial_t v(t,x)\big|_{t=0} = 0$, whenever the last one exists. In particular, $\partial_\tau W_{\ell,\sigma}(t,\tau) \geq 0$ for all $t \geq 0$ and $\tau > 0$, and

$$\int_0^\infty \partial_\tau(-W_{\ell,\sigma}(t,\tau))d\tau = 1,$$

for all $t \geq 0$.

Now, we are in position to prove our main result.

Theorem (5.4)**.** *Let $(k, \ell) \in (\mathcal{PC})$, see Definition 2.12 below, such that ℓ is 1-regular and θ-sectorial with $\theta < \frac{\pi}{2}$, see Definition 2.14 and 2.13 below. Then, the fundamental solution $U_k(t,x)$ of (1.10) can be represented as follows*

$$U_k(t,x) = -\int_0^\infty P(\tau,x)\partial_\tau W_{\ell,\sigma}(t,\tau)d\tau, \quad t > 0,\ x \in \mathbb{R},$$

where $P(t,x)$ is the probability density function of the classical telegraph process, σ is an arbitrary positive constant, and $W_{\ell,\sigma}$ is the propagation function associated to the function ℓ and σ, see Definition 2.16 below.

Using the previous Theorem, we will show, by means of several examples, that the distribution of some concrete stochastic processes appears in the context of subdiffusion theory. To this end, we prove the following proposition.

Proposition (5.4)**.** *Let $(k, \ell) \in (\mathcal{PC})$, see Definition 2.12 below. The fundamental solution of (1.10) can be interpreted as the distribution of a process of the form $T(|\mathcal{W}(t)|)$, $t > 0$, where $T(t)$ is the telegraph process and $|\mathcal{W}(t)|$ is a random time whose law is $2V_{2k}(t,\tau)$ with $t > 0$ and $\tau > 0$, where $V_{2k}(\cdot,\cdot)$ is the solution to the nonlocal equation*

$$\partial_t^2(k * k * (u(\cdot,x) - \delta_0(x))(t) = \sigma^2 \partial_x^2 u(t,x), \quad t > 0,\ x \in \mathbb{R},$$

subject to the initial conditions

$$u(0,x) = \delta_0(x), \quad \text{and} \quad \partial_t u(t,x)\big|_{t=0} = 0,$$

where the condition imposed in the first derivative must be considered only when exists.

Finally, we present some relevant examples that appear in that chapter.

Example (5.10)**.** *Let $n \in \mathbb{N}$. If $\alpha = \frac{1}{2^{n+1}}$ and $\sigma^2 = 2^{\frac{1}{2^n}-2}$ then distribution of $X_\alpha(t)$ coincides with the distribution of the process*

$$T(|B_1(|B_2(|\cdots|B_n(|B_{n+1}(t)|)|)\cdots|)|), \quad t > 0,$$

where $T(t)$ is the classical telegraph process and $B_1, B_2 \cdots B_{n+1}$ are $(n+1)$ independent Brownian motions.

Example (5.11)**.** *If $\alpha = \frac{1}{3}$ and $\sigma^2 = 1$, then the distribution of $X_\alpha(t)$ coincides with the distribution of the process $T(|A(t)|)$, $t > 0$, where $T(t)$ is the classical telegraph process and $A(t)$ is a stochastic process whose law is given by $\frac{3}{2}V(t,\tau)$, for $t > 0$ and $\tau > 0$, where $V(\cdot,\cdot)$ is the fundamental solution of the following evolution equation*

$$\partial_t v(t,x) = \partial_x^3 v(t,x), \quad t > 0, \; x \in \mathbb{R}.$$

Example (5.13)**.** *Let $n \in \mathbb{N}$. If $\alpha = \frac{1}{3 \cdot 2^n}$ and $\sigma^2 = 2^{\frac{1}{2^n}-1}$ then distribution of $X_\alpha(t)$ coincides with the distribution of the process*

$$T(|B_1(|B_2(|\cdots|B_{n-1}(|A(t)|)|)\cdots|)|), \quad t > 0,$$

where $T(t)$ is the classical telegraph process, $B_1, B_2 \cdots B_{n-1}$ are $n-1$ independent Brownian motions and $A(t)$ is the stochastic process described in Example 5.11.

Organization of the Thesis

In the following chapter, we give some preliminary concepts. In Chapter 3, we give physical foundations for our research. We explain that in order to account for dispersion in a rigid linear isotropic medium, in [95, Chapter II, Section 9.5] the author proposed to modify the classical constitutive relations and generated a new model for the classical telegraph equation. Also, we will give an explanation of the concepts and the subsequent formulation of this general telegraph equation and its relationship with the classical telegraph equation and its non-local version. Finally, we will give a description of the different models that can be obtained from the proposed general equation. In Chapter 4, we will answer question **(Q1)**. In this chapter, we are going to study the asymptotic behavior of the variance of the process associated to the fundamental solution of (1.10), and then, we will explain a method to construct new examples such that the variance has an ultra-slow growth behavior. In Chapter 5, we will answer questions **(Q2)**, **(Q3)** and **(Q4)**. We will study, in detail, the moments of the process $X_k(t)$. In particular, we will show that the moments satisfy the so-called Carleman condition, and consequently, the distribution of the process is uniquely determined by them. Moreover, we will find a representation of the moment generating function of the distribution. In addition, we will prove that the probability density function can be constructed via subordination from the distribution of the classic telegraph process.

Preliminaries

In this chapter, we collect some preliminary material that we need later on in order to develop this thesis and other classic results for a greater completeness of this chapter.

2.1 Completely Monotonic Functions

2.1.1 Definitions and Basic Properties

Definition 2.1. ***(Laplace transform).*** *Let $f \in L_{1,loc}(\mathbb{R}_+)$ and $z \in \mathbb{C}$. If the improper integral*

$$\mathscr{L}(f;\lambda) = \widehat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt,$$

exists in the sense of Bochner, we call it the Laplace integral or the Laplace transform of the function f at the point λ. On the other hand, given a measure μ on the half-line $[0,\infty)$, the Laplace transform for μ is defined by

$$\mathscr{L}(\mu;\lambda) = \widehat{\mu}(\lambda) = \int_{[0,\infty)} e^{-\lambda t}\mu(dt),$$

as long as it exists.

***Remark* 2.1.** *In the previous definition $\mathscr{L}(f,\lambda) = \mathscr{L}(\mu,\lambda)$ if $\mu(dt)$ denotes the measure $f(t)dt$.*

The following definition presents a class of functions with a particular interest for us since their Laplace transform is very well understood.

Definition 2.2. ***(Completely Monotonic).*** *A function $f : (0,+\infty) \to \mathbb{R}$ is a completely monotonic function if f is of class C^∞ and*

$$(-1)^n f^{(n)}(z) \geq 0, \quad \forall n \in \mathbb{N}_0, \quad \text{and} \quad z > 0.$$

The family of all completely monotonic functions will be denoted by $(\mathcal{CM})$. The following theorem is known as *Bernstein's Theorem*, this theorem gives the characterization of completely monotonic functions. We can find a brief proof in [33].

Theorem 2.1. ***(Bernstein's Theorem).*** *A C^∞-function $f : (0,\infty) \to \mathbb{R}$ is completely monotonic if, and only if, there is a non-decreasing function $b : \mathbb{R}_+ \to \mathbb{R}_+$, such that*

$$f(z) = \int_0^\infty e^{-zt}db(t), \quad z > 0.$$

Normalizing $b(t)$ by $b(0) = 0$ and $b(t)$ left-continuous, $b(t)$ is uniquely determined by f. Moreover,

$$(-1)^n f^{(n)}(z) = \int_0^\infty e^{-zt}t^n db(t), \quad z > 0, n \in \mathbb{N}_0,$$

and

$$(-1)^n f^{(n)}(0^+) = \int_0^\infty t^n db(t), \quad n \in \mathbb{N}_0.$$

Summarizing completely monotonic functions are Laplace transforms of positive measures supported on $\mathbb{R}_+$. In particular, Bernstein's Theorem shows that every $f \in (\mathcal{CM})$ admits a holomorphic extension to $\mathbb{C}_+$.

The next two corollaries of Bernstein's Theorem are taken from [103, Corollary 1.6 and Corollary 1.7 resp.]

Corollary 2.1. *The set $(\mathcal{CM})$ of completely monotonic functions is a convex cone, i.e.*

$$sf_1 + tf_2 \in (\mathcal{CM}), \quad \text{for all} \quad s,t \geq 0 \quad \text{and} \quad f_1, f_2 \in (\mathcal{CM}),$$

which is closed under multiplication, i.e.

$$z \to f_1(z)f_2(z) \quad \text{is in} \quad (\mathcal{CM}), \quad \text{for all} \quad f_1, f_2 \in (\mathcal{CM}),$$

and under pointwise convergence:

$$(\mathcal{CM}) = \overline{\{\mathfrak{L}\mu : \mu \text{ is a finite measure on } [0,+\infty)\}}$$

(The closure is taken with respect to pointwise convergence).

Corollary 2.2. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of completely monotonic functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ exists for all $x \in (0,\infty)$. Then $f \in (\mathcal{CM})$ and $\lim_{n\to\infty} f_n^{(k)}(x) = f^{(k)}(x)$ for all $k \in \mathbb{N}_0$ locally uniformly in $x \in (0,\infty)$.*

Proposition 2.1. *Suppose $f(x)$ is completely monotonic function with measure μ. Then*

$$\lim_{x\to\infty} x^n f^{(n)}(x) = 0, \quad \forall n \geq 1.$$

Proof. First of all, we notice that $u^n e^{-u} \leq (n+1)^n e^{-n} e^{-u/(n+1)}$ for all $n \geq 1$ and $u > 0$, this is because the function $f_k(x) = x^k e^{-xk/(k+1)}$ is bounded from above by $(k+1)^k e^{-k}$ on $(0,\infty)$ for all $k \geq 1$. And by the Bernstein's Theorem 2.1, we obtain

$$\begin{aligned}|x^n f^{(n)}(x)| =& x^n \int_{[0,\infty)} e^{-xt} t^n \mu(dt) = \int_{(0,\infty)} e^{-xt} (xt)^n \mu(dt) \\ &\leq \int_{(0,\infty)} (n+1)^n e^{-n} e^{-xt/(n+1)} \mu(dt) = (n+1)^n e^{-n} \left(f\left(\frac{x}{n+1}\right) - \mu(\{0\}) \right).\end{aligned}$$

As $\lim_{x\to\infty} f(x)) = \mu(\{0\})$, letting x to approaches infinity, we get the conclusion for all $n \geq 1$. □

We refer to Gripenberg, Londen and Staffans [39, Theorem 5.2.6] for the following theorem, which summarises properties of the Laplace transform of completely monotonic and locally integrable kernels.

Theorem 2.2. *Let $a \in L_{1,loc}(\mathbb{R}_+)$ be completely monotonic. The Laplace transform $\widehat{a}$ has the following properties:*

1. *The Laplace transform $\widehat{a}$ has a holomorphic extension to $\mathbb{C}\backslash(-\infty, 0)$ via*

$$\widehat{a}(\lambda) = \int_0^\infty \frac{d\beta(t)}{\lambda + t},$$

   where β is the uniquely determined function from Bernstein's Theorem.

2. *The Laplace transform $\widehat{a}(x)$ is real and non-negative for $x > 0$.*

3. *We have $\lim_{x\to\infty} \widehat{a}(x) = 0$.*

4. *$Im\ \widehat{a}(\lambda) \leq 0$ for all $Im\ \lambda > 0$.*

Moreover, $a \in L_1(\mathbb{R}_+)$ if and only if $\lim_{x\to 0} |\widehat{a}(x)| < \infty$.

Now we introduce a version of Karamata-Feller Tauberian theorem. The proof can be found in [11, Section 1.7, Chapter I] or [34, Chapter XIII].

Definition 2.3. *Let $\varrho \in \mathbb{R}$. We say that a function $L : (0,\infty) \to (0,\infty)$ is a regularly varying function at infinity of index ϱ if for every fixed $x > 0$ we have that*

$$\lim_{t\to\infty} \frac{L(tx)}{L(t)} = x^{\varrho}.$$

This class of functions will be denoted by $\mathcal{RV}_{\infty}^{\varrho}$. The class of regularly varying at infinity functions of index $\varrho = 0$ is also known as slowly varying functions and it will be denoted by $\mathcal{SV}_{\infty}$.
The function L is called **regularly varying at zero** *if the function $t \mapsto L(t^{-1})$ belongs to the class $\mathcal{RV}_{\infty}^{\varrho}$. In such a case, we write $L \in \mathcal{RV}_{0}^{\varrho}$. The class $\mathcal{RV}_{0}^{0}$ is known as slowly varying at zero functions and it will be denoted by $\mathcal{SV}_{0}$.*

***Remark* 2.2.** *Let $\varrho \in \mathbb{R}$ and $F \in \mathcal{RV}_{\infty}^{\varrho}$. It follows from [11, Theorem 1.4.1] that there exists $L \in \mathcal{SV}_{\infty}$ such that $F(x) = x^{\varrho}L(x)$ for $x > 0$.*

***Remark* 2.3.** *Let $F : (0,\infty) \to (0,\infty)$ be a regularly varying function at infinity of index ϱ. It follows from [11, Theorem 1.4.1] that there is a slowly varying function $L : (0,\infty) \to (0,\infty)$ such that $F(x) = x^{\varrho}L(x)$ for $x > 0$.*

Theorem 2.3. ***(Karamata-Feller's theorem).*** *Let $L_1, L_2 : (0,\infty) \to (0,\infty)$ be slowly varying functions. Let $\beta > 0$ and $w : (0,\infty) \to \mathbb{R}$ be a monotone function whose Laplace transform $\widehat{w}(\lambda)$ exists for all $\lambda \in \mathbb{C}_+$. Then*

$$\widehat{w}(\lambda) \sim \frac{1}{\lambda^{\beta}} L_1(\lambda), \text{ as } \lambda \to \infty, \quad \text{if and only if} \quad w(t) \sim \frac{t^{\beta-1}}{\Gamma(\beta)} L_1\left(\frac{1}{t}\right), \text{ as } t \to 0^+,$$

and

$$\widehat{w}(\lambda) \sim \frac{1}{\lambda^{\beta}} L_2\left(\frac{1}{\lambda}\right), \text{ as } \lambda \to 0^+, \quad \text{if and only if} \quad w(t) \sim \frac{t^{\beta-1}}{\Gamma(\beta)} L_2(t), \text{ as } t \to \infty,$$

where $\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1}e^{-x}dx$, $Re(\alpha) > 0$. Here the limits for λ are on the positive real axis and the notation $f(t) \sim g(t)$ as $t \to t_*$ means that $\lim_{t\to t_*} f(t)/g(t) = 1$.

The following is an essential definition for the development of this thesis.

Definition 2.4. ***(Convolution).*** *For $k \in L_{1,loc}(\mathbb{R}_+)$ and $f \in L_{1,loc}(\mathbb{R}_+; X)$, with X a Banach space, then we define the convolution between k and f for $t \in \mathbb{R}_+$ by*

$$(k * f)(t) = \int_0^t k(t-s)f(s)ds,$$

and the integral is understood in the sense of Bochner.

Proposition 2.2. *Let $k, h \in L_{1,loc}(\mathbb{R}_+)$ and $f \in L_1(\mathbb{R}_+; X)$. Then*

1. *$(k * f)(t)$ exists for almost all $t \in \mathbb{R}_+$ and $k * f \in L_{1,loc}(\mathbb{R}_+; X)$.*
2. *$h * (k * f) = (h * k) * f$ a.e.*

Proof. These results may be deduced from the vector-valued version of Fubini Theorem, see [5, Theorem 1.1.9]. ☐

Theorem 2.4. ***(Young Convolution Inequality).*** *Suppose $f \in L^p(\mathbb{R}^d)$ and $g \in L^r(\mathbb{R}^d)$ and $p, q, r \in [1,\infty]$ and $p, q \le r$ such that*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1,$$

we have that

$$||f * g||_r \le ||f||_p ||g||_q.$$

For more details about Young convolution inequality, see [115, Chapitre III] or [53, 98] and references therein.

Definition 2.5. ***(Abscissa of convergence).*** *Let f be a function and $\widehat{f}$ be its Laplace transform. The abscissa of convergence is defined by*

$$abs(f) := inf\left\{Rez : \widehat{f}(z) exists\right\}.$$

Proposition 2.3. *Let $k \in L_{1,loc}(\mathbb{R}_+)$, $f \in L_1(\mathbb{R}_+; X)$, $\lambda \in \mathbb{C}$, and suppose that $Re\,\lambda > max(abs(|k|, abs|f|)$. Then $\widehat{(k * f)}(\lambda)$ exists and $\widehat{(k * f)}(\lambda) = \widehat{k}(\lambda)\widehat{f}(\lambda)$.*

2.1.2 Existence and Operational Properties of the Laplace Integral

The importance of Laplace integral in applications to differential equations lies in that it transforms the analytic operations of differentiation, integration, and convolution into algebraic operations of multiplication. In the first place, we give a condition for the existence of the Laplace transform of a function f.

The proof of the following proposition is taken from [5, Proposition 1.4.1].

Proposition 2.4. *Let $f \in L_{1,loc}(\mathbb{R}_+)$. Then the Laplace integral $\widehat{f}$ converges if $Re\,\lambda > abs(f)$ and diverges if $Re\,\lambda < abs(f)$.*

Proof. Clearly, $\widehat{f}$ does not exist if $Re\,\lambda < abs(f)$. For $\lambda_0 \in \mathbb{C}$ define $G_0(t) := \int_0^t e^{-\lambda_0 s} f(s)ds$ $(t \geq 0)$. Then, for all $\lambda \in \mathbb{C}$ and $t \geq 0$, integration by parts gives

$$\int_0^t e^{-\lambda s} f(s)ds = \int_0^t e^{-(\lambda-\lambda_0)s} e^{-\lambda_0 s} f(s)ds$$
$$= e^{-(\lambda-\lambda_0)t} G_0(t) + (\lambda - \lambda_0) \int_0^t e^{-(\lambda-\lambda_0)s} G_0(s)ds.$$

If $\widehat{f}(\lambda_0)$ exists, then G_0 is bounded. Moreover, it follows from the previous equality that $\widehat{f}(\lambda)$ exists if $Re\,\lambda > Re\,\lambda_0$ and

$$\widehat{f}(\lambda) = (\lambda - \lambda_0) \int_0^\infty e^{-(\lambda-\lambda_0)s} G_0(s)ds, \quad (Re\,\lambda > Re\,\lambda_0).$$

This shows that $\widehat{f}(\lambda)$ exists if $Re\,\lambda > abs(f)$. □

***Remark* 2.4.** *If $\widehat{f}$ converges for all $\lambda \in \mathbb{C}$, then $abs(f) = -\infty$. If the domain of convergence is empty, then $abs(f) = \infty$. A function f is called Laplace transformable if $abs(f) < \infty$.*

Theorem 2.5. ***(Uniqueness theorem).*** *Let $f, g \in L_{1,loc}(\mathbb{R}_+; X)$ with $abs(f) < \infty$ and $abs(g) < \infty$, and let $\lambda_0 > max(abs(f), abs(g))$. Suppose that $\widehat{f}(\lambda) = \widehat{g}(\lambda)$ whenever $\lambda > \lambda_0$. Then $f(t) = g(t)$ a.e.*

Proposition 2.5. *Let $f \in L_{1,loc}(\mathbb{R}_+)$ and let $F(t) = \int_0^t f(s)ds$. If $Re\,\lambda \geq 0$ and $\widehat{f}(\lambda)$ exists, then $\widehat{F}(\lambda)$ exists and $\widehat{F}(\lambda) = \dfrac{\widehat{f}(\lambda)}{\lambda}$.*

Proof. This is immediate from [5, Proposition 1.3.1] with $k(t) = 1$. □

Corollary 2.3. *Let $f : \mathbb{R}_+ \to X$ be absolutely continuous and differentiable a.e.. If $Re\,\lambda > 0$ and $\widehat{f'}(\lambda)$ exists, then $\widehat{f}(\lambda)$ exists and $\widehat{f'}(\lambda) = \lambda\widehat{f}(\lambda) - f(0^+)$.*

2.2 Bernstein and Stieltjes Functions

The class of *Bernstein functions* is closely related to that of completely monotonic functions.

2.2.1 Bernstein Functions

Definition 2.6. ***(Bernstein function).*** *A function $f:(0,\infty)\to\mathbb{R}$ is called a Bernstein function if $c\in C^{\infty}(0,\infty)$, $f(t)\geq 0$ for all $t\geq 0$ and f' is completely monotonic. The class of Bernstein functions will be denoted by $(\mathcal{BF})$.*

Theorem 2.6. *A function $f:(0,\infty)\to\mathbb{R}$ is a Bernstein function if, and only if, it admits the representation*

$$f(z)=a+bz+\int_{(0,\infty)}(1-e^{-zt})\mu(dt), \tag{2.1}$$

where $a,b\geq 0$ and μ is a measure on $(0,\infty)$ satisfying $\int_{(0,\infty)}(1\wedge t)\mu(dt)<\infty$.

For a proof of the following structural characterization of Bernstein functions, we refer to [103, Theorem 3.6]

Theorem 2.7. *Let f be a positive function on $(0,\infty)$. Then the following assertions are equivalent.*

1. *$f\in(\mathcal{BF})$.*
2. *$g\circ f\in(\mathcal{CM})$ for every $g\in(\mathcal{CM})$.*
3. *$e^{-uf}\in(\mathcal{CM})$ for every $u>0$.*

Corollary 2.4.
1. *The set $(\mathcal{BF})$ is a convex cone, closed under pointwise limits and under composition.*
2. *For all $f\in(\mathcal{BF})$, the function $\lambda\to\dfrac{f(\lambda)}{\lambda}$ is in $(\mathcal{CM})$.*
3. *Let $f_1,f_2\in(\mathcal{BF})$ and $\alpha,\beta\in(0,1)$ be such that $\alpha+\beta\leq 1$. Then $\lambda\to f_1(\lambda^{\alpha})f_2(\lambda^{\beta})$ is again a Bernstein function.*

By virtue of the representation of Bernstein function obtained in Equation (2.1), it is convenient to introduce the following definition.

Definition 2.7. ***(Creep function).*** *A function $k:(0,\infty)\to\mathbb{R}$ is called a creep function if $k(t)$ is non-negative, non-decreasing and concave. A creep function $k(t)$ has the standard form*

$$k(t)=k_0+k_{\infty}t+\int_0^t k_1(\tau)d\tau,\quad t>0, \tag{2.2}$$

where $k_0=k(0+)\geq 0, k_{\infty}=\lim_{t\to\infty}k(t)/t$ and $k_1(t)=\dot{k}(t)-k_{\infty}$ is non-negative and non-increasing with $\lim_{t\to\infty}k_1(t)=0$. The class of creep functions will be denoted by $(\mathcal{CF})$.

The next class of function has been used throughout the literature in many branches of mathematics but under various names such as *Nervanlinna functions* in complex interpolation theory, *operator monotone functions* in functional analysis, or *class (S)* in the Russian literature on complex functions theory.

Definition 2.8. ***(Completely Bernstein Function).*** *A Bernstein function f is said to be a completely Bernstein Function if its Lévy measure μ in 2.1 has a completely monotonic density $m(t)$ with respect to Lebesgue measure,*

$$f(\lambda)=a+b\lambda+\int_{(0,\infty)}(1-e^{-\lambda t})m(t)dt. \tag{2.3}$$

We will use $(\mathcal{CBF})$ to denote the collection of all completely Bernstein functions.

***Remark* 2.5.** *Thus Theorem 2.6 and the Equation (2.2) state that $\varphi\in(\mathcal{BF})$ iff $\varphi(\lambda)=\lambda\widehat{dk}(\lambda)$ for some $k\in(\mathcal{CF})$. Further, the complete Bernstein functions φ are represented as $\varphi(\lambda)=\lambda\widehat{dk}(\lambda)$ with some $k\in(\mathcal{BF})$. We have the inclusions $(\mathcal{CBF})\subset(\mathcal{BF})\subset(\mathcal{CF})$.*

The following Theorem establishes these statements and another equivalence, see [103, Theorem 6.2].

Theorem 2.8. *Suppose that f is a non-negative function on $(0,\infty)$. Then the following conditions are equivalent.*

1. $f \in (\mathcal{CBF})$.
2. *The function $\lambda \to f(\lambda)/\lambda$ is in $(\mathcal{S})$, see definition 2.9.*
3. *There exists a Bernstein function g such that*
$$f(\lambda) = \lambda^2 \mathscr{L}(g;\lambda), \quad \lambda > 0.$$
4. *f has an analytic continuation to $\mathbb{H}^{\uparrow}$ such that $Im\ f(z) \geq 0$ for all $z \in \mathbb{H}^{\uparrow}$ and such that the limit $f(0^+) = \lim_{(0,\infty)\ni\lambda\to 0} f(\lambda)$ exists and is real.*
5. *f has an analytic continuation to the cut complex plane $\mathbb{C}\backslash(-\infty,0]$ such that $Im\ z \cdot Im\ f(z) \geq 0$ and such that the limit $f(0^+) = \lim_{(0,\infty)\ni\lambda\to 0} f(\lambda)$ exists and is real.*
6. *f has an analytic continuation to $\mathbb{H}^{\uparrow}$ which is given by*
$$f(z) = a + bz + \int_{(0,\infty)} \frac{z}{z+t}\sigma(dt),$$
where $a, b \geq 0$ are non-negative constants and σ is a measure on $(0,\infty)$ such that
$$\int_{(0,\infty)} (1-t)^{-1}\sigma(dt) < \infty.$$

2.2.2 Stieltjes Functions

The next type of functions is a very important subclass of completely monotonic functions. These play a central role in the study of completely Bernstein functions.

Definition 2.9. ***(Stieltjes funtion).*** *A (non-negative) Stieltjes function is a function $f : (0,\infty) \to [0,\infty)$ which can be written in the form*

$$f(\lambda) = \frac{a}{\lambda} + b + \int_{(0,\infty)} \frac{1}{\lambda+t}\sigma(dt),$$

where $a, b \geq 0$ are non-negative constants and σ is a measure on $(0,\infty)$ such that $\int_{(0,\infty)}(1+t)^{-1}\sigma(dt) < \infty$. We denote the family of all Stieltjes Functions by $(\mathcal{S})$.

For a proof of the following Theorem, we refer to [103, Theorem 2.2].

Theorem 2.9.

1. *Every $f \in (\mathcal{S})$ is of the form*
$$f(\lambda) = \mathscr{L}(a\cdot dt;\lambda) + \mathscr{L}(b\cdot\delta_0(dt);\lambda) + \mathscr{L}(\mathscr{L}(\sigma;t)dt;\lambda)$$
for the measure appearing in Definition 2.9. In particular, $(\mathcal{S}) \subset (\mathcal{CM})$ consists of all completely monotonic functions having a representation measure with completely monotonic density on $(0,\infty)$.
2. *The set $(\mathcal{S})$ is a convex cone and closed under pointwise limits.*

Proposition 2.6. *$f \in (\mathcal{CBF})$, $f \not\equiv 0$, if, and only if, the function $f^*(\lambda) := \lambda/f(\lambda)$ is in $(\mathcal{CBF})$.*

Proof. If $f \in (\mathcal{CBF})$ we may use the representation, see [103, Remark 6.4], and write

$$\frac{f(z)}{z} = \frac{a}{z} + b + \int_{(0,\infty)} \frac{1}{z+t}\sigma(dt), \tag{2.4}$$

where a, b and σ are defined as in Definition 2.9. Since

$$Imz \cdot Im\frac{1}{z+t} = Imz \cdot \frac{-Imz}{|z+t|^2} = \frac{-(Imz)^2}{|z+t|^2} \leq 0,$$

we see that $f(z)/z$ maps $\mathbb{H}^{\uparrow}$ into $\mathbb{H}^{\downarrow}$. As $1/z$ switches the upper and lower half-planes, $z/f(z) = (z^{-1}f(z))^{-1}$ maps $\mathbb{H}^{\uparrow}$ into itself. Further, $\lambda/f(\lambda) \in (0,\infty)$ for $\lambda > 0$, and since $a = \lim_{\lambda\to 0^+} f(\lambda)$, we have

$$\lim_{\lambda\to 0^+} \frac{\lambda}{f(\lambda)} = \begin{cases} 0, & \text{if } a \neq 0, \\ \frac{1}{\lim_{\lambda\to 0^+}\frac{f(\lambda)}{\lambda}} = \frac{1}{b+\int_{(0,\infty)}\frac{\sigma(dt)}{t}} \in [0,\infty), & \text{if } a = 0. \end{cases}$$

Theorem 2.8 shows that $\lambda/f(\lambda)$ is in $(\mathcal{CBF})$. Conversely, if $g(\lambda) = \lambda/f(\lambda), \lambda \in (0,\infty)$, is a completely Bernstein function, we can apply the just established result to this function and get

$$(\mathcal{CBF}) \ni \frac{\lambda}{g(\lambda)} = \frac{\lambda}{\frac{\lambda}{f(\lambda)}} = f(\lambda).$$

□

The following Theorem is a characterization of Stieltjes functions. Note that this theorem enables us to transfer all statements for completely Bernstein functions to Stieltjes functions.

Theorem 2.10. *A function $f \not\equiv 0$ is a completely Bernstein function if, and only if, $1/f \not\equiv 0$ is an Stieltjes function. In other words*

$$(\mathcal{CBF})^* = \{f : 1/f \in (\mathcal{S})^*\} \quad \text{and} \quad (\mathcal{S})^* = \{g : 1/g \in (\mathcal{CBF})^*\}.$$

($(\mathcal{CBF})^*, (\mathcal{S})^*$ refer to the not identically vanishing elements of $(\mathcal{CBF})$ and $(\mathcal{S})$ resp.)

Proof. If $f \in (\mathcal{CBF})^*$ Proposition 2.6 shows that $\lambda \to \lambda/f(\lambda)$ is also a completely Bernstein function. As such, $z/f(z)$, $z \in \mathbb{C}\backslash(-\infty,0]$, has a representation of the form 2.1 dividing by z we see that $1/f(z)$ is an Stieltjes function. Conversely, it is obvious from the definition of Stieltjes functions and Proposition 2.6. □

The following Corollary is an immediate consequence of Theorem 2.10 combined with Theorem 2.8.

Corollary 2.5. *Let g be a positive function on $(0,\infty)$. Then g is a Stieltjes function if, and only if, $g(0^+)$ exists in $[0,\infty]$ and g extends analytically to $\mathbb{C}\backslash(-\infty,0]$ such that $Im\ z \cdot Im\ g(z) \le 0$, i.e. g maps $\mathbb{H}^{\uparrow}$ to $\mathbb{H}^{\downarrow}$ and vice versa.*

The set $(\mathcal{S})$ plays pretty much the same role for $(\mathcal{CBF})$ as do the set $(\mathcal{CM})$ for the set $(\mathcal{BF})$. The following theorem is the $(\mathcal{CBF})$-analogue of Theorem 2.7. For proof, we refer to [103, Theorem 7.5].

Theorem 2.11. *Let f be a positive function on $(0,\infty)$. Then the following assertions are equivalent.*

1. *$f \in (\mathcal{CBF})$.*
2. *$g \circ f \in (\mathcal{S})$ for every $g \in (\mathcal{S})$.*
3. *$\frac{1}{u+f} \in (\mathcal{S})$ for every $u > 0$.*

The next two Corollaries show how these classes of functions, $(\mathcal{S})$ and $(\mathcal{CBF})$, operate on each other. For a proof, we refer to [103, Corollary 7.6 and Corollary 7.9].

Corollary 2.6. *The set $(\mathcal{CBF})$ is a convex cone, closed under pointwise limits and under composition.*

Corollary 2.7. *Let $f, g \in (\mathcal{CBF})$ be a completely Bernstein functions and let $s, t \in (\mathcal{S})$ be Stieltjes functions. Then*

1. *$f \circ s \in (\mathcal{S})$,*
2. *$s \circ f \in (\mathcal{S})$,*
3. *$f \circ g \in (\mathcal{CBF})$,*
4. *$s \circ t \in (\mathcal{CBF})$.*

The following theorem is helpful for the development of our theory. For proof, we refer to [103, Theorem 7.10].

Theorem 2.12. *The sets $(\mathcal{CBF})$ and $(\mathcal{S})$ are both logarithmically convex, i.e.*

$$\text{for all } f_1, f_2 \in (\mathcal{CBF}) \text{ and for all } \alpha \in (0,1) \text{ then } f_1^\alpha f_2^{1-\alpha} \in (\mathcal{CBF}),$$

$$\text{for all } f_1, f_2 \in (\mathcal{S}) \text{ and for all } \alpha \in (0,1) \text{ then } f_1^\alpha f_2^{1-\alpha} \in (\mathcal{S}).$$

2.3 Scalar Volterra Equations and Kernels

We want to summarise basic definitions and properties concerning scalar Volterra equations and the associated kernels. We will introduce the concept of scalar and integral resolvent for linear scalar Volterra equations and their representations of solutions. A linear scalar Volterra equation is an equation of the type

$$u(t) + \eta(u * \ \ell)(t) = f(t), \quad t > 0, \tag{2.5}$$

where $\eta \in \mathbb{C}$, $\ell \in L_{1,loc}(\mathbb{R}_+, \mathbb{R})$ and $f : \mathbb{R}_+ \to \mathbb{C}$ is a given function. These types of equations have been deeply studied, see [25, 26, 39], and are essential for the treatment of linear Volterra equations.

Definition 2.10. ***(Scalar and Integral Resolvent).*** *Let $\eta \in \mathbb{C}$ and $\ell \in L_{1,loc}(\mathbb{R}_+)$. The solution $s_\eta \colon \mathbb{R}_+ \to \mathbb{C}$ of the scalar Volterra equation*

$$s_\eta(t) + \eta(s_\eta * \ell)(t) = 1, \quad t > 0, \tag{2.6}$$

is called scalar resolvent associated to ℓ. The solution $r_\eta \colon \mathbb{R}_+ \to \mathbb{C}$ of the scalar Volterra equation

$$r_\eta(t) + \eta(r_\eta * \ell)(t) = \ell(t), \quad t > 0, \tag{2.7}$$

is called integral scalar resolvent associated to ℓ.

We point out that for every $\eta \in \mathbb{C}$ there exists $\lambda_\eta \geq 0$ such that the Laplace transforms of $s(t,\eta)$ and $r(t,\eta)$ are given by

$$\widehat{s}(\lambda,\eta) = \frac{\widehat{k}(\lambda)}{\lambda\widehat{k}(\lambda) + \eta} = \frac{1}{\lambda(1 + \eta\widehat{\ell}(\lambda))}, \quad \lambda > \lambda_\eta, \tag{2.8}$$

and

$$\widehat{r}(\lambda,\eta) = \frac{1}{\lambda\widehat{k}(\lambda) + \eta} = \frac{\widehat{\ell}(\lambda)}{1 + \eta\widehat{\ell}(\lambda)}, \quad \lambda > \lambda_\eta. \tag{2.9}$$

***Remark* 2.6.** *It follows from [95, Proposition 4.5] that the condition $(\mathcal{PC})$ implies that $\varphi(\lambda) = \lambda\widehat{k}(\lambda)$ is a Bernstein function. Thus, it follows that if $\eta \in \mathbb{R}_+$, then $\lambda_\eta = 0$.*

We know from [39, Chapter 2, Theorem 3.1] that for each locally integrable kernel $\ell \in L_{1,loc}(\mathbb{R}_+)$ there is a unique and locally integrable scalar integral resolvent $r_\eta \in L_{1,loc}(\mathbb{R}_+)$ solution of (2.7). Moreover, it follows from [39, Chapter 2, Theorem 3.5] that the solution of (2.5) is given by

$$u(t) = f(t) - \eta(r_\eta * f)(t), \quad t > 0.$$

In particular, this implies the existence and uniqueness of the scalar resolvent $s_\eta \in L_{1,loc}(\mathbb{R}_+)$ whenever $\ell \in L_{1,loc}(\mathbb{R}_+)$ and the following relation between the scalar resolvent and integral resolvent:

$$s_\eta(t) = 1 - \eta(r_\eta * 1)(t), \quad t > 0. \tag{2.10}$$

Both r_η and s_η have been successfully applied in the study of asymptotic decay rates of some non-local in time differential equations, see [56, 93, 113].

Definition 2.11. ***(Completely positive).*** *A kernel $\ell \in L_{1,loc}(\mathbb{R}_+)$ is called completely positive if for all $\eta \in \mathbb{R}_+$ the functions r_η and s_η are non-negative on $\mathbb{R}_+$. The class of completely positive functions will be denoted by $(\mathcal{CP})$.*

There is another equivalent definition of completely positive functions, for example it is known from Clemént and Nohel [26] that the kernel $\ell \in L_{1,loc}(\mathbb{R}_+)$ is completely positive if ℓ is non-negative, non-increasing and $\log \ell$ is convex on $\mathbb{R}_+$. Moreover, in [95], the definition extends to completely positive measures.

An essential class of completely positive functions is the pair $(k, \ell) \in (\mathcal{PC})$, which means that the following condition is satisfied.

Definition 2.12. ***(Condition*** $(\mathcal{PC})$***).*** *That the function* $k \in L_{1,loc}(\mathbb{R}_+)$ *is non-negative and non-increasing, and there exists a kernel* $\ell \in L_{1,loc}(\mathbb{R}_+)$ *such that* $k * \ell = 1$ *in* $(0, \infty)$*. In this case we write* $(k, \ell) \in (\mathcal{PC})$*.*

These kernels are also called *Sonine kernels*, see [100], and they have been successfully used to study integral equations of first kind in the space of Hölder continuous, Lebesgue and Sobolev functions, see [22]. Further, the condition $k * \ell = 1$ is also called the *Sonine condition* and based on this, it is possible to deduce some general properties of the kernels, see [99]. Note that these properties are satisfied for any pair of completely positive kernels.

Remark **2.7.** *Using the fact that* $(k, \ell) \in (\mathcal{PC})$*, we note that convolving* (2.7) *with* k*, we have that*

$$(k * r_\eta)(t) + \eta(1 * r_\eta)(t) = 1, \quad t > 0.$$

This shows that the pair $(r_\eta, \eta 1 + r_\eta)$ *is a Sonine pair of functions, see [99].*

Lemma 2.1. *Let* (k, ℓ) *be a pair of type* $(\mathcal{PC})$*. Then the following relations hold for all* $t > 0$*:*

1. $tk(t)\ell(t) \leq 1$,
2. $k(t) \int_0^t \ell(\tau)d\tau \leq 1, \quad \ell(t) \int_0^t k(\tau)d\tau \leq 1$,
3. $k(t) \int_0^t \ell(\tau)d\tau + \ell(t) \int_0^t k(\tau)d\tau \geq 1$,
4. $\int_0^t k(t) \int_0^t \ell(\tau)d\tau \geq t$,
5. $\lim_{t\to 0^+} k(t) = \lim_{t\to 0^+} \ell(t) = +\infty$,
6. $\lim_{t\to 0^+} tk(t) = \lim_{t\to 0^+} t\ell(t) = 0$.

A very important example of $(k, \ell) \in (\mathcal{PC})$ is given by the pair $(g_{1-\alpha}, g_\alpha)$ with $\alpha \in (0, 1)$, where g_β is the standard notation of the Equation (1.11).

The following proposition contains several different characterizations of completely positives measures; for the proof, we refer to [95, Proposition 4.5].

Proposition 2.7. *Suppose* $c \in BV_{loc}(\mathbb{R}_+)$ *is Laplace transformable and such that* $\widehat{dc}(\lambda) > 0$ *for all* $\lambda > 0$*. Then the following assertions are equivalent:*

1. $dc \in (\mathcal{CP})$.
2. $\varphi(\lambda) = \dfrac{1}{\widehat{dc}(\lambda)} \in (\mathcal{BF})$.
3. $\psi_\tau(\lambda) = exp\left(\dfrac{\tau}{\widehat{dc}(\lambda)}\right) \in (\mathcal{CM})$ *for every* $\tau > 0$.
4. $\varphi_\mu = \dfrac{\widehat{dc}(\lambda)}{1 + \mu\widehat{dc}(\lambda)} \in (\mathcal{CM})$*, for every* $\mu > 0$.
5. $s(t, \mu)$ *is positive and non-increasing w.r.t.* $t > 0$*, for every* $\mu > 0$*, where* s *is the solution of* $s(t; \mu) + \mu \int_0^t s(t - \tau; \mu)dc(\tau) = 1, \quad t, \mu > 0$.

Corollary 2.8. *Let* dc *be a completely positive measure, and let* $\phi_\mu(\lambda) = \lambda\widehat{s}(\lambda; \mu) = \varphi(\lambda)/(\mu + \varphi(\lambda))$*, with* $\lambda, \mu > 0$*. Then* $\phi_\mu \in (\mathcal{BF})$*.*

***Remark* 2.8.** *Let $(k,\ell) \in (\mathcal{PC})$. If we set also $k \in (\mathcal{CM})$, then the application $\varphi : \lambda \to \lambda\widehat{k}(\lambda)$ is a completely Bernstein function. Indeed, it is easy to see that $\lambda\widehat{k}(\lambda) = \lambda^2\frac{\widehat{k}(\lambda)}{\lambda} = \lambda^2\mathscr{L}(1 * k;\lambda)$, and since $k \in (\mathcal{CM})$, then $1 * k \in (\mathcal{BF})$. Then by Theorem 2.8 φ is a completely Bernstein function. On the other hand, if $(k_1,\ell_1),(k_2,\ell_2) \in (\mathcal{PC})$ and $k_1,k_2 \in (\mathcal{CM})$ according Theorem 2.12, the function $\lambda \to \sqrt{\lambda\widehat{k_1}(\lambda)z\widehat{k_2}(\lambda)}$ is a completely Bernstein function.*

We introduce the following notation.

Notation 2.1. *If the functions k and ℓ satisfy the conditions $(k,\ell) \in (\mathcal{PC})$ and $k \in (\mathcal{CM})$, we write $(k,\ell) \in (\mathcal{PCM})$.*

2.4 The Subordination Principle in the sense of Prüss

The class of completely positive kernels plays a prominent role in the theory of vector-valued Volterra equations, see Definition 2.15, and appears in applications quite naturally. This class of kernels, its properties, and associated creep functions appear in our work through the so-called *subordination*. By means of the principle of subordination, it is possible to construct a new resolvent from a given one, e.g., from a C_0-semigroup or from a cosine family. The new resolvent can be explicitly represented in terms of the given one and of the propagation function associated with a completely positive kernel. This representation is particularly useful for the understanding of the regularity and the asymptotic behavior of the resolvent.

Let us first state some definitions that will be used in Chapter 5.

Definition 2.13. ***(Sectorial).*** *Let $k \in L_1, loc(\mathbb{R}_+)$ be of subexponential growth and suppose $\widehat{k}(\lambda) \neq 0$ for all $Re\,\lambda > 0$. Then, k is called sectorial with angle $\theta > 0$ (or merely θ-sectorial) if*

$$|\arg\widehat{k}(\lambda)| \leq \theta \quad \text{for all } Re\,\lambda > 0.$$

Here $\arg\widehat{k}(\lambda)$ is defined as the imaginary part of a fixed branch of $\log\widehat{k}(\lambda)$, and θ need not be less than π. If a is sectorial, we always choose that branch of $\log\widehat{k}(\lambda)$, which gives the smallest angle θ; in particular, in case $\widehat{k}(\lambda)$ is real for λ, we choose the principal branch.

Definition 2.14. ***(Regular).*** *Let $k \in L_1, loc(\mathbb{R}_+)$ be of subexponential growth, and $n \in \mathbb{N}$. $k(t)$ is called n-regular if there is a constant $c > 0$ such that*

$$|\lambda^m\widehat{k}^{(m)}(\lambda)| \leq c|\widehat{k}(\lambda)| \quad \text{for all} \quad Re\,\lambda > 0, \quad 0 \leq m \leq n.$$

Observe that any n-regular kernel $k(t)$, $n \geq 1$ has the property that $\widehat{k}(\lambda)$ has no zeros in the open right halfplane.

***Remark* 2.9.** *We know from [95, Pag. 69] that for a n-regular kernel k, $n \in \mathbb{N}$ arbitrary, its Laplace transform $\widehat{k}$ has no zeros in the open right half-plane. Futhermore, each completely monotonic kernel is n-regular for all $n \in \mathbb{N}$, see [95, Proposition 3.3]. On the other hand, according to [95, Pag. 70], if $k(t)$ is real-valued and 1-regular then $k(t)$ is $\frac{c\pi}{2}$-sectorial, with c as in Definition 2.14. Moreover, we can conclude that*

$$(k,\ell) \in (\mathcal{PC}) \text{ and } \ell \text{ log-convex } \Rightarrow \ell \text{ is 1-regular and } \frac{c\pi}{2}\text{-sectorial.}$$

The theorem of Bernstein on completely monotonic functions will play a fundamental role in this subsection. Let $b \in BV_{loc}(\mathbb{R}_+)$ be non-decreasing such that $\int_0^\infty db(t)e^{-\lambda t} < \infty$ for each $\lambda > 0$, and let dc be a completely positive measure. Then $f(\lambda) = \widehat{db}(\lambda)$ is completely monotonic, and $\varphi(\lambda) = \dfrac{1}{\widehat{dc}(\lambda)}$ is a Bernstein function. By Proposition 2.7, $f \circ \varphi$ is completely monotonic. Hence there is $a \in BV_{loc}(\mathbb{R}_+)$ non-decreasing such that

$$\widehat{da}(\lambda) = f(\varphi(\lambda)) = \widehat{db}\left(\frac{1}{\widehat{dc}(\lambda)}\right), \quad \lambda > 0. \tag{2.11}$$

This is the so-called *subordination principle* for completely positive measures.

Definition 2.15. ***(Volterra equation).*** *Let X be a complex Banach space, $\mathcal{A}$ a closed linear unbounded operator in X with dense domain $\mathcal{D}(A)$, $u_0 : \mathbb{R}^d \to \mathbb{R}$, $g : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ and $b \in L_{1,loc}(\mathbb{R}_+)$ a scalar kernel $\not\equiv 0$. A Volterra equation is an equation of the form*

$$u(t,x) + (b * \mathcal{A}u(\cdot,x))(t) = u_0(x) + g(t,x), \quad t > 0, x \in \mathbb{R}^d.$$

Consider now the Volterra equation

$$u(t) = f(t) + \int_0^t b(t-\tau)\mathcal{A}u(\tau)d\tau, \quad t > 0, \tag{2.12}$$

where $\mathcal{A}$ is closed, linear and densely defined, the kernel $b \in L_{1,loc}(\mathbb{R}_+)$ is such that $\int_0^\infty |b(t)|e^{-\beta t}dt < \infty$, and let $c(t) \in (\mathcal{CP})$. Define

$$\widehat{a}(\lambda) = \widehat{b}\left(\frac{1}{\widehat{c}(\lambda)}\right), \quad \lambda > 0. \tag{2.13}$$

The Volterra equation

$$v(t) = g(t) + \int_0^t a(t-\tau)\mathcal{A}v(\tau)d\tau, \quad t > 0, \tag{2.14}$$

is then called *subordinate* to (2.12) via $c(t)$. According to [95, Theorem 4.1], Equation (2.14) admits a resolvent whenever (2.12) does; this is the *general subordination principle for the resolvent.* The next lemma presents a composition rule involving three Bernstein functions, and this is very useful in applications, see [95, Lemma 4.3]

Lemma 2.2. *Suppose $a, b, c \in (\mathcal{BF})$. Then there is a Bernstein function e such that*

$$\widehat{e}(\lambda) = \widehat{a}(\lambda)\widehat{dc}\left(\frac{\widehat{a}(\lambda)}{\widehat{b}(\lambda)}\right), \quad \lambda > 0.$$

Moreover, $e_0 = a_0(c_0 + \widehat{c}(a_0/b_0))$ for $a_0 > 0$, $e_0 = 0$ otherwise.

Remark **2.10.** *As an application of this above Lemma, note that $\widehat{a}(\lambda)^{1-\alpha}\widehat{b}(\lambda)^{\alpha}$ is the Laplace transform of a Bernstein function, whenever $a, b \in (\mathcal{BF})$ and $\alpha \in [0,1]$.*

For the following proposition, we need to introduce the notion of *propagation function.* Let dc be a completely positive measure, $\varphi(\lambda) = \dfrac{1}{\widehat{dc}(\lambda)}$ its associated Bernstein function, and let $k \in BV_{loc}(\mathbb{R}+)$ be the creep function such that $\varphi(\lambda) = \lambda\widehat{dk}(\lambda)$, $\lambda > 0$, see Definition 2.7. By Proposition 2.7, the functions $\psi_\tau(\lambda) = exp(-\tau\varphi(\lambda))$ are completely monotonic with respect to $\lambda > 0$, for each fixed $\tau \geq 0$, and bounded $e^{-\tau\varphi(0^+)} = e^{-\tau k_0}$. From Bernstein's Theorem, see 2.1, it follows that there are unique non-decreasing functions $\omega(\cdot;\tau) \in BV(\mathbb{R}_+)$, normalized by $\omega(0;\tau) = 0$, and left-continuous, such that

$$\widehat{\omega}(\lambda;\tau) = \frac{\psi_\tau(\lambda)}{\lambda}, \quad \lambda > 0, \tau \in \mathbb{R}_+. \tag{2.15}$$

Note that $\omega(\cdot;\tau)$ enjoys the semigroup property

$$\begin{aligned} \int_0^t \omega(t-s;\tau_1)d\omega(s;\tau_2) =& \omega(t;\tau_1+\tau_2), \quad t, \tau_1, \tau_2 \geq 0, \\ \omega(t;0) =& e_0(t), \quad t \geq 0. \end{aligned} \tag{2.16}$$

where $e_0(t)$ denotes the Heaviside function.

Definition 2.16. ***(Propagation function).*** *The function $\omega(t;\tau)$ defined above is called the propagation function associated with the completely positive measure dc.*

For the proof of the following proposition, we refer to [95, Proposition 4.9 and Corollary 4.5].

Proposition 2.8. *Let $\mathcal{A}$ be a closed linear densely defined operator in a Banach space X, and $b, c \in L_{1,loc}(\mathbb{R}_+)$, with $c(t) \in (\mathcal{CP})$, such that $\int_0^\infty |b(t)|e^{-\beta t}dt < \infty$ for some $\beta \in \mathbb{R}$ and let $\omega(t;\tau)$ be the propagation function associated with $c \in L_{1,loc}(\mathbb{R}_+)$. Then the resolvents $S_a(t)$ and $S_b(t)$ of (2.14) and (2.12), respectively, are related by*

$$S_a(t) = -\int_0^\infty S_b(\tau) d_\tau \omega(t:\tau), \quad t > 0. \tag{2.17}$$

***Remark* 2.11.** *Given the equations*

$$\underbrace{u(t,x) + (\ell * (-\Delta)u(\cdot,x))(t) = u_0,}_{\text{Volterra equation}} \qquad \underbrace{u(t,x) + \big(1 * (-\Delta)u(\cdot,x)\big)(t) = u_0\,.}_{\text{Heat equation}}$$

Note that $\mathcal{L}(1;\lambda) = \dfrac{1}{\lambda}$ and $\widehat{\ell}(\lambda) = \dfrac{1}{\frac{1}{\widehat{\ell}(\lambda)}}$. Then if $\ell \in (\mathcal{CP})$ we have $\widehat{c}(\lambda) = \dfrac{1}{\widehat{\ell}(\lambda)}$ is a Bernstein function, and by Bernstein's Theorem, see 2.1, we know that the solution s_μ, from $s_\mu + \mu(c * s_\mu)(t) = 1$, is completely monotonic w.r.t. μ, where $s_\mu(t) = -\int_0^\infty e^{-\tau\mu}\omega(t;\tau)d\tau$. Then by Proposition 2.8, using the equation (2.17), we have

$$S_{Volterra}(t,x) = -\int_0^\infty S_{Heat}(\tau,x)\omega(t,\tau)d\tau, \quad t > 0, x \in \mathbb{R}^d.$$

So we point out that a Volterra equation with a completely positive kernel is subordinate to the heat equation.

2.5 Probability and Random Variables

In this section we refer to [18, 24, 91] for the definitions and properties mentioned.

Remember that a **random experiment** is a process by which we observe something uncertain. After the experiment, the result of the random experiment is known. An **Outcome** is a result of a random experiment, and the set of all possible outcomes is called the **sample space**. We assign a probability measure $P(A)$ to an event A. This is a value between 0 and 1 that shows how likely the event is. Probability theory is based on some axioms that act as the foundation for the theory.

Definition 2.17. *Axioms of Probability*

Axiom 1: For any event A, $P(A) \geq 0$.

Axiom 2: Probability of the sample space S is $P(S) = 1$.

Axiom 3: If $A_1, A_2, A_3, ...$ are disjoint events, then $P(A_1 \cup A_2 \cup A_3 ...) = P(A_1) + P(A_2) + P(A_3) + ...$

Definition 2.18. ***(Random Variables).*** *A random variable X is a function from the sample space to the real number*

$$X : S \to \mathbb{R}.$$

The range of a random variable X is the set of possible values of X. We will say that X is a discrete random variable if its range is countable.

Note that if X is a discrete random variable with range R_X, we can list the elements in R_X. In other words we can write

$$R_X = \{x_1, x_2, x_3, ...\}\,.$$

Definition 2.19. *Let X be a discrete random variable with range R_X. The function*

$$P_X(x_k) = P(X = x_k), \qquad \text{for } k = 1, 2, 3, ...,$$

*is called the probability mass function **(PMF)** of X.*

***Remark* 2.12.** *The PMF is a probability measure that gives us probabilities of the possible values for a random variable.*

Definition 2.20. ***(Cumulative distribution function).*** *The cumulative distribution function* ***(CDF)*** *of random variable X is defined as*

$$F_X(x) = P(X \leq x), \quad \text{for all } x \in \mathbb{R}.$$

Definition 2.21. ***(Continuous random variables).*** *A random variable X with CDF $F_X(x)$ is said to be continuous if $F_X(x)$ is a continuous function for all $x \in \mathbb{R}$.*

We will also assume that the CDF of a continuous random variable is differentiable almost everywhere in $\mathbb{R}$. Note that to determine the distribution of a discrete random variable we can either provide its PMF or CDF. For a continuous random variable, the CDF is well-defined so we can provide the CDF. However, the PMF does not work for continuous random variables because, for a continuous random variable, $P(X = x) = 0$ for all $x \in \mathbb{R}$.

Definition 2.22. ***(Probability density function).*** *Consider a continuous random variable X with an absolutely continuous CDF $F_X(x)$. The function $f_X(x)$ defined by*

$$f_X(x) = \frac{dF_X(x)}{dx} = F'_X(x), \quad \text{if } F_X(x) \text{ is differentiable at } x,$$

is called the probability density function ***(PDF)*** *of X.*

Let us summarize some properties of the PDF.

Proposition 2.9. *Consider a continuous random variable X with PDF $f_X(x)$. We have that*

1. *$f_X(x) \geq 0$ for all $x \in \mathbb{R}$.*
2. *$\int_{-\infty}^{\infty} f_X(u)du = 1$.*
3. *$P(a \leq x \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(u)du$.*
4. *More generally, for a set A, $P(X \in A) = \int_A f_X(u)du$.*

Definition 2.23. ***(Expectation).*** *Let X be a continuous random variable. The expected value E, or the expectation, of X is defined as*

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)dx.$$

Theorem 2.13. ***(Law of the unconscious statistician).*** *Let X be a continuous random variable, g a function with domain D and $X \subseteq D$. The Law of the unconscious statistician* ***(LOTUS)*** *for X establishes that*

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) f_X(x)dx.$$

Definition 2.24. ***(Variance).*** *The Variance of the random variable X is defined as*

$$Var(X) = E\left[(X - E(X))^2\right].$$

Definition 2.25. ***(Moments).*** *The nth moment of a random variable X is defined to be $E\left[X^n\right]$. The nth central moment of X is defined to be $E\left[(X - EX)^n\right]$.*

Definition 2.26. ***(Moment generating function).*** *The Moment generating function* ***(MGF)*** *of a random variable X is a function $M_X(z)$ defined as*

$$M_X(z) = E\left[e^{zX}\right].$$

We say that MGF of X exists, if there exists a positive constant R_0 such that $M_X(z)$ is finite for all $z \in (-R_0, R_0)$.

Theorem 2.14. *Suppose that the MGF, $M_X(z)$, is well-defined on a neighbourhood of the origin, say $(-R_0, R_0)$. Then X has finite absolute moments of all orders. Moreover, $M_X(z)$ admits an absolutely convergent Taylor expansion*

$$\boldsymbol{M}(t,z) = \sum_{n=0}^{\infty} \frac{z^n\, E\left[X^n\right]}{n!},\ |z| < R_0. \tag{2.18}$$

***Remark* 2.13.** *Note that we can obtain all moments of X^n from its MGF:*

$$E\left[X^n\right] = \frac{d^n}{dz^n} M_X(z)|_{z=0}.$$

Also, it is well-known that the cumulative distribution function and the probability density function can often be found by applying the inverse Laplace transform to the moment generating function, provided that this function converges.

Definition 2.27. ***(Stochastic process).*** *A random process or a stochastic process is a collection of random variables usually indexed by time. Also we say that a random process is a random function of time.*

***Remark* 2.14.** *Note that for a random process $\{X(t), t \in J\}$, for some index J, the mean function is defined as $E\left[X(t)\right]$ from J to $\mathbb{R}$.*

Definition 2.28. *Two random processes are equivalent in distribution if they have the same finite dimensional distribution.*

Definition 2.29. ***(Markov process).*** *The stochastic process $X(t)$ is said to be a* Markov process *provide that for any $t, s \geq 0$ and $j \in \mathbb{R}$*

$$P\left\{X_{t+s} = j \,|\, X_u;\ u \leq t\right\} = P\left\{X_{t+s} = j \,|\, X_t\right\}.$$

Chapter 3

Physics model

To understand the complexity of the model, we start this chapter with a historical review about difficulties to achieve transoceanic communication in its beginnings, for more details we refer the readers to [6, Page 121] and the references therein.

3.1 Historical review

At the dawn of 20^{th} century, Sir William Thomson, known as Lord Kelvin, was the first person to explain the electrical theory behind the operation of land based telegraph lines allowing him to conceive a new system for successful undersea telegraphy. This cable established the first electrical communications link between, the so-called, old (Great Britain) and new worlds (United States). The American entrepreneur, Cyrus Field, spearheaded efforts which spanned more than 14 years and ultimately cost over US12 million, approximately US150 million currently. Key scientific and engineering contributions were made by Sir William Cooke, Sir Charles Wheatstone, Michael Faraday and Professor James Clerk Maxwell. Samuel F.B. Morse and William Thomson each spent significant time working onboard the ships to address the major engineering problems facing the project.

The attempts in 1857 and 1858 involved the two largest warships in the world, the *USS Niagara* and the *HMS Agamemnon*. The 1857 expedition ended in failure when the cable snapped as it was being released from onboard the ship. A second attempt in 1858 also failed. In a third attempt during July and August 1858, the *Niagara* and the *Agamemnon* successfully laid the first operable transatlantic cable. Unfortunately, the cable failed after only three weeks. The failure was ultimately attributed to high signaling voltages which “burned out” the line. A fourth attempt was made in 1865. The new cable was nearly four times as bulky and almost twice as heavy as the 1858 cable based on design changes recommended by Professor Thomson and the Chief Engineer Charles Bright. The 1865 cable again broke in mid-ocean, leaving the expedition with another failure. A fifth attempt in 1866 finally brought success. The expedition that year not only succeeded in laying a fresh cable, but also recovered the broken 1865 section from the ocean floor and attached it to a new portion laid in 1866, completing a second line. The 1866 cable operated successfully until 1872, and the restored 1865 cable operated until 1877. By then, both cables had been rendered obsolete. They were functionally replaced by a new 1873 Anglo-American Telegraph Company cable made by Telcon and the 1874 Direct United States Cable Company cables made by Siemens Brothers. The cable project demanded new technologies and new science to overcome unforeseen obstacles and the rigors of the north Atlantic.

As early as 1848, Thomson recognized the enormous possibilities for electrical science and its potential use of applied mathematical reasoning. The key lay in Thomson’s unique combination of Fourier’s mathematical methods and his analogies between the theories of heat transfer and electrical impulses. The scientists involved with the cable project soon realized that submarine cables behaved differently than land lines. Defining, measuring and accounting for differences in electrical resistance in underwater cables became the chief obstacle to successful long distance submarine telegraphy.

In 1823, Sir Francis Ronalds first found that electrical signals were retarded when passing through an insulated wire, or core, laid under ground. The same effect was observed in wire immersed in water.

Michael Faraday concluded that the retardation was caused by induction between the electricity in the wire and the earth or water surrounding it. As the cable's core receives a charge from a battery, the electricity induces an opposite charge in the water as it travels along; and, as the two charges attract each other, the "exciting" charge is restrained. The resulting speed of a signal through the conductor is "thereby retarded by its own making".

In overland lines the current traverses the wire suddenly, like a bullet, and at its full strength, so that if the current be sufficiently strong these instruments will be worked at once, and no time will be lost. But it is quite different on submarine cables. There the current is slow and varying. It travels along the copper wire in the form of a wave or undulation, and is received feebly at first, then gradually rising to its maximum strength, and finally dying away again as slowly as it rose. This is owing to the phenomenon of induction, very important in submarine cables, but almost entirely absent in land lines. Now the electricity sent into this wire induce electricity of an opposite kind to itself in the sea-water outside, and the attraction set up between these two kinds "holds back" the current in the wire, and retards its passage to the receiving station.

Thomson recognized that the speed of a telegraph signal was limited by both its capacity and resistance. He calculated that the speed decreased as the square of the cable length increased for any given diameter of the core conductor. His computations confirmed the "capacitance" (the amount of stored electric charge for a given electric potential) of a cable. He found that a cable surrounded by an insulator residing in a conducting medium (salt water) acted as a form of condenser; interacting and exchanging electrical potential with the surrounding medium. Thomson devised his "doctrine of squares" to define the relationship. His concept soon came to be known as Thomson's "Law of Squares".

According to the Law of Squares, a cable two miles long would have four times the retardation in signal strength of a cable that was one mile long, and the strength of the signal would therefore be only $\frac{1}{4}$ as strong. Consequently, for any given cable, signaling speed is inversely proportional to the square of the cable length, when holding the capacitance and resistance constant. Thomson used these relationships to define signal arrival curves in which he could compute the arriving signal current after any interval of time following the signal transmission based on the battery voltage, the cable resistance, the cable capacity, and length of the cable. Thomson had succeeded in mathematically proving Faraday's initial theory.

The diameter of the 1865 and 1866 copper cores and the surrounding insulation were increased almost three-fold over the original 1858 cable design to allow the weak currents to flow more easily. Thomson analyzed electro-magnets and concluded that similar iron cores with winding lengths proportional to the squares of their linear dimensions produced equal intensities of magnetic fields when they were supplied with equal currents. Thomson's conclusion successfully identified the relationship between iron cores and copper conducting windings. Knowledge of this relationship enabled the production of electromagnets which could yield similar magnetic intensity despite using different sizes of iron cores and different lengths of windings. Thomson had defined the parameters under which all future electromagnets would be manufactured.

3.2 Physics foundations of the model

In this section, we will present an excerpt from article [94] where the physical and mathematical foundations for the formulation of the general equation of the non-local telegraph in time are presented. Although this part is not used directly in the development of the following chapters, it gives physical support to our work, so it is important to mention it.

It is well-known, see e.g., [52, Chapter 6], that the macroscopic electrodynamic fields in a medium at rest in the considered inertial system, are governed by *Maxwell´s equations*

$$\mathcal{B}_t + \nabla \times \mathcal{E} = 0, \quad \nabla \cdot \mathcal{B} = 0 \tag{3.1}$$

$$\mathcal{D}_t - \nabla \times \mathcal{H} + \mathcal{J} = 0, \quad \nabla \cdot \mathcal{D} = \rho \tag{3.2}$$

where $\mathcal{E}$ denotes the electric field, $\mathcal{H}$ the magnetic field, $\mathcal{B}$ the magnetic induction, $\mathcal{D}$ the electric induction, $\mathcal{J}$ the free current and ρ the free charge, see [107, Chapter 2]. Clearly, (3.1) and (3.2) are indeterminate and constitutive equations have to be supplemented, which specify the electrodynamic properties of the medium in question. For the simplest medium- the vacuum. the relations are given by

$$\mathcal{B} = \mu_0 \mathcal{H}, \quad \mathcal{D} = \varepsilon_0 \mathcal{E}, \quad \mathcal{J} = 0, \tag{3.3}$$

where $\varepsilon_0, \mu_0 > 0$ are well-known fundamental physical constants which are connected with the speed of light in vacuum c_0 by the relation

$$\varepsilon_0 \mu_0 = c_0^{-2}.$$

The next simplest medium is the rigid linear isotropic dielectric, where the constitutive relations are defined by

$$\mathcal{B} = \mu \mathcal{H}, \quad \mathcal{D} = \varepsilon \mathcal{E}, \quad \mathcal{J} = 0, \tag{3.4}$$

where ε and μ are material constants with $\varepsilon \geq \varepsilon_0$ and $\mu > 0$. These constitutive relations do not account for the dielectric losses observed, when the medium is placed in a rapidly varying electromagnetic field. This defect cannot be removed by the introduction of a conductivity $\sigma > 0$ and replacing the last equation in (3.4) by a relation of the form $\mathcal{J} = \sigma \mathcal{E}$ (Ohm law), i.e.

$$\mathcal{B} = \mu \mathcal{H}, \quad \mathcal{D} = \varepsilon \mathcal{E}, \quad \mathcal{J} = \sigma \mathcal{E}, \tag{3.5}$$

More precisely, if one considers periodic fields $\mathcal{E}(t) = \mathcal{E}_0 e^{i\omega t}$ and $\mathcal{D}(t) = \mathcal{D}_0 e^{i\omega t}$, one still observes relations of the form $\mathcal{D}_0 = \mathcal{E}_0 \widehat{\varepsilon}$ for each frequency ω. However, $\widehat{\varepsilon}$ will be complex in general and will depend of ω. This phenomenon is known as *electromagnetic dispersion* in the physical literature.
To account for dispersion in a rigid linear isotropic medium the following constitutive relations have been proposed in [95, Capter II, Section 9.5]

$$\mathcal{B} = d\mu * \mathcal{H} \tag{3.6}$$

$$\mathcal{D} = d\varepsilon * \mathcal{E} \tag{3.7}$$

$$\mathcal{J} = d\sigma * \mathcal{E} \tag{3.8}$$

where $\mu, \varepsilon, \sigma \in BV(\mathbb{R}_+)$ are given material functions. Not much seems to be known about general material functions. However, following [95, Capter II, Section 9.5] one can guess that $\varepsilon(t)$ should be a bounded creep function. It is also reasonable to expect that $\sigma(t)$ is a bounded creep function as well. In the paramagnetic case, we have that $\mu(\infty) > \mu_0$, while $\mu_0 > \mu(\infty)$ holds for diamagnetic media. Hence, in the former case $\mu(t)$ can be considered a bounded creep function, while in the latter case $\mu_0 - \mu(t)$ should be such.

3.2.1 Circuit model of a transmission line

A transmission line is a one-dimensional system, in which voltage and currents depend on time t and on a longitudinal coordinate that always be indicated by z. The state variables of a system are then $v(t,z)$ and $i(t,z)$. The standard circuit model in a infinitesimal segment of line z is shown in the following figure:

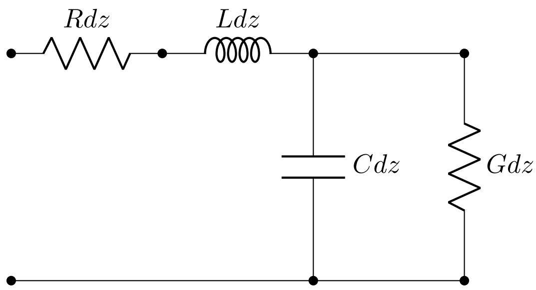

Figure 3.1: Circuit Model.

Our aim is to obtain the relations between the current i and voltage v into the different devices resistors R and G, inductor L, and the capacitor C. To this end, we use the presentation (for the classic case) in [107, Chapter 3]. To follow our analysis, we considered the following figure:

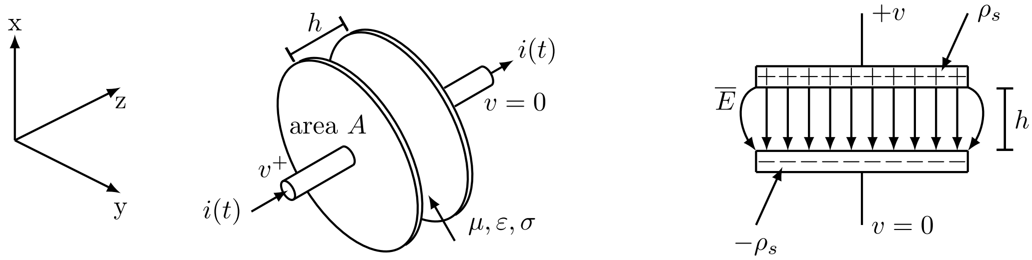

Figure 3.2: Resistor

The resistor illustrated in Figure 3.2 is comprised of two parallel perfectly conducting end-plates between which is placed a medium of conductivity σ, permittivity ε, permeability μ and thickness h; the two end plates and the medium all have a constant cross-sectional area $A\left[m^2\right]$ in the xy-plane. Let us assume a static voltage v exists across the resistor R, and current i flows through it.

Resistors. Since the conductivity σ is uniform within walls parallel to $\widehat{z}$, these constraints are satisfied by a static uniform electric field $\mathcal{E} = \widehat{z}\mathcal{E}_0$ everywhere within the conducting medium, which would be charge-free since weare assuming that $\mathcal{E}$ is non-divergent. Therefore, following [107, Chapter 3, formula 3.1.2], we note that the voltage v is given by the product of the distance h and the z-coordinate of the electric field $\mathcal{E}$, that is

$$v(t) = \int_0^h \mathcal{E} \cdot \widehat{z} dz = h\mathcal{E}_0(t), \quad t > 0.$$

The total current i flowing is the integral of $\mathcal{J} \cdot \widehat{z}$ over the device cross-section A, see [107, Chapter 3, formula 3.1.4]. Using (3.8) we have

$$i(t) = \int\int_A \mathcal{J} \cdot \widehat{z} dxdy = \int\int_A d\sigma * \mathcal{E} \cdot \widehat{z} dxdy = \frac{A}{h}(d\sigma * v)(t). \tag{3.9}$$

We point out that considering $d\sigma = \sigma_0 d\delta_0$ (Dirac measure) we recover the Ohm law, this is, $v = Ri$, with $R = \dfrac{h}{A\sigma_0}$, see [107, Chapter 3.1.1].

Capacitors. Following [107, Chapter 3, formula 3.1.9], we have that the local charge Q on the positive end plate of area A at time t is given by

$$Q(t) = A\rho(t) = A\nabla \cdot \mathcal{D} = Ad\varepsilon * \mathcal{E}_0 = \frac{A}{h}(d\varepsilon * v)(t),$$

by (3.2) and (3.7). On the other hand, since the charge Q of the positive plate is the time integral of the current i into it, we have

$$\frac{A}{h}(d\varepsilon * v)(t) = Q(t) = \int_{-\infty}^{t} i(s)ds,$$

and consequently

$$i(t) = \frac{A}{h}\partial_t(d\varepsilon * v)(t). \tag{3.10}$$

Inductors. To continue our analysis, we consider [107, Figure 3.2.1 and Figure 3.3.2, pages 72-73]

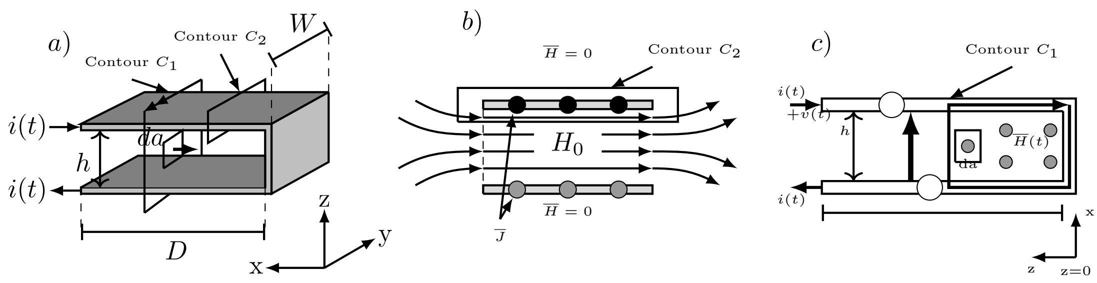

Figure 3.3: Inductors

According to the left-hand law, we have for the magnetic field $\mathcal{H}$ that $\mathcal{H}(t) = H_0 \cdot \widehat{y}$. The voltage $v(t)$ across the terminals of the inductor illustrated in Figure 3.3 can be found using the integral form of (3.1), the constitutive equation (3.6) and $\mathcal{H} = H_0 \cdot \widehat{y} = \dfrac{i(t)}{W}$, that is

$$\oint_C \mathcal{E}ds = -\partial_t \int_A d\mu * \mathcal{H}da = -\partial_t \int_A d\mu * H_0 \cdot \widehat{y}da = -\frac{Dh}{W}\partial_t(d\mu * i)(t) = -v(t)$$

where $z = D$, at the inductor terminals. Hence

$$v(t) = \frac{Dh}{W}\partial_t(d\mu * i)(t). \tag{3.11}$$

Now, we are in position to derive a non-local in time telegraph equation. To this end, we considered a circuit for transmission line model, see Figure 3.1. Since we have two resistors R and G, we define $d\sigma_R$ for R and $d\sigma_G$. If we assume that $d\sigma_G$ is invertible in the convolution sense, that is, there exists a measure $d\overline{\sigma}_R$ such that $d\sigma_R * d\overline{\sigma}_R = 1$, then (3.9) can be written as

$$v(t) = \frac{h}{A}\partial_t(d\overline{\sigma}_R * i)(t). \tag{3.12}$$

It is worthwhile to note the condition of invertibility of $d\sigma_R$ in the convolution sense is not too restrictive and there are several example where it is fulfilled. Further, this condition is fundamental to apply the Kirchhoff´s circuit law. So, the total tension on the circuit is

$$\partial_z v(t,z) + \frac{Dh}{W}\partial_t(d\mu * i)(t) + \frac{h}{A}(d\overline{\sigma}_R * i)(t) = 0 \tag{3.13}$$

and the total current

$$\partial_z i(t,z) + \frac{A}{h}\partial_t(d\varepsilon * v)(t) + \frac{A}{h}(d\sigma_G * v)(t) = 0. \tag{3.14}$$

We point out that (3.13) and (3.14) are the non-local in time versions of (3.16) and (3.17) respectively. Following the same ideas of [90, Chapter 3, Section 3.3] and without any considerations about regularity of the functions v and i, the equations (3.16) and (3.17) can be combined to obtain the following equation

$$\kappa\partial_t^2(d\varepsilon * d\mu * u) + \partial_t([d\overline{\sigma}_R * d\varepsilon + \kappa d\mu * d\sigma_G] * u) + d\overline{\sigma}_R * d\sigma_G * u - \partial_z^2 u = 0 \tag{3.15}$$

where $\kappa := \dfrac{AD}{W}$ and u could be defined as v or i, since it is the same equation for both.

3.2.2 Different models

Since (3.15) is given by means of measures, we can obtain several telegraphic models choosing these measures appropriately. Next, we give some examples that illustrate the different kind of telegraph models that we can deduce from (3.15).

(1) The telegraph equation. To show this model, at first we considered $v(t,z)$ and $i(t,z)$ respectively to be the voltage and the current at the point of the cable with coordinate z at instant t in a long cable consisting of two parallel wires stretched along the z-axis. In 1893, Heaviside in [46, Section 201] established that the relation between $v(t,z)$ and $i(t,z)$ can be described by the following differential equations

$$\partial_x v(t,x) + L\partial_t i(t,x) + Ri(t,x) = 0, \quad t>0, x\in\mathbb{R} \tag{3.16}$$

$$\partial_x i(t,x) + C\partial_t v(t,x) + Gv(t,x) = 0, \quad t>0, x\in\mathbb{R} \tag{3.17}$$

where R denotes resistance, L self-inductance, G the leak-conductance and C capacitance; these quantities are measured per unit length of the cable. Following [90, Chapter 3, Section 3.3], we point out that these two equations can combined to obtain an equation of the form

$$LC\ \partial_t^2 u + (RC+GL)\ \partial_t u + GR\ u - \partial_x^2 u = 0, \quad t>0,\ x\in\mathbb{R}, \tag{3.18}$$

where the function u could be defined as the voltage v or the current i.

Let us notice now that if we set

$$d\mu = \mu_0 d\delta_0, \quad d\varepsilon = \varepsilon_0 d\delta, \quad d\overline{\sigma}_R = \frac{1}{\sigma_0}d\delta_0, \quad d\sigma_G = \frac{1}{\sigma_1}d\delta_0,$$

in (3.15) and define

$$L = \frac{\mu_0 Dh}{W}, \quad C = \frac{A\varepsilon_0}{h}, \quad R = \frac{h}{A\sigma_0}, \quad G = \frac{A}{\sigma_1 h}$$

where $A, D, W, h, \mu_0, \varepsilon_0, \sigma_0$ and σ_1, are defined above, the equation (3.15) takes the form

$$LC\ \partial_t^2 u + (RC+GL)\ \partial_t u + GR\ u - \partial_x^2 u = 0, \quad t>0,\ x\in\mathbb{R},$$

which is exactly the equation (3.18).

We recall that if we assume high frequency in the model, then the effect of the resistor is negligible and consequently we can take $d\sigma_G \equiv 0$. This consideration leads to

$$LC\partial_t^2 u + RC\partial_t u - \partial_x^2 u = 0, \quad t>0, x\in\mathbb{R},$$

which is the equation (1.4). This shows that from (3.15) it can be obtained a very interesting equation even when the effect of the resistor is negligible.

(2) Hyperbolic telegraph equation with memory. Let $k \in L_{1,loc}(\mathbb{R}_+)$ and set

$$d\mu = \mu_0 d\delta_0, \quad d\varepsilon = \varepsilon_0 d\delta, \quad d\overline{\sigma}_R = \frac{1}{\sigma_0} k(s)ds, \quad d\sigma_G \equiv 0,$$

in (3.15), where μ_0, ε_0 and σ_0 are positive constants. Then the general non-local telegraph equation takes the form

$$\partial_t^2 u(t,x) + \eta \partial_t (k * u(\cdot,x))(t) - \nu \partial_x^2 u(t,x) = 0, \quad t > 0, x \in \mathbb{R}, \tag{3.19}$$

where $\eta = \dfrac{W}{\sigma_0 \mu_0 DA}$ and $\nu = \dfrac{W}{\mu_0 \varepsilon_0 DA}$ where A, D and W, are defined in the same manner of the model (3.15).

The equation (3.19) has been successfully used to model anomalous diffusion in some viscoelastic materials such as polymers, porous materials, among others, see [17] and references therein. However, this equation never has been considered as a telegraph processes until [94].

(3) The telegraph equation with one dynamic. Let $k \in L_{1,loc}(\mathbb{R}_+)$ and set

$$d\mu = \mu_0 k(t)dt, \quad d\varepsilon = \varepsilon_0 k(t)dt, \quad d\overline{\sigma}_R = \frac{1}{\sigma_0} d\delta_0, \quad d\sigma_G \equiv 0,$$

in (3.15), where μ_0, ε_0 and σ_0 are positive constants. Then the general non-local telegraph equation takes the form

$$\partial_t^2 (k * k * u(\cdot,x))(t) + \eta \partial_t (k * u(\cdot,x))(t) - \nu^2 \partial_x^2 u(t,x) = 0, \quad t > 0, x \in \mathbb{R},$$

where $\eta = \dfrac{1}{\sigma_0 \mu_0 \kappa} > 0$ and $\nu^2 = \dfrac{1}{\mu_0 \varepsilon_0 \kappa} > 0$, and $\kappa = \dfrac{DA}{W}$ where A, D and W, are defined in the same manner of the model (3.15), which is the equation (1.10).

In this case, we note that the measures $d\mu$ and $d\varepsilon$ are induced by the same function k, up the multiplication by a constant. Therefore, it clear that (1.10) depends only on one function k. To due this, we say that the equation is a one dynamic non-local telegraph equation.

(4) The telegraph equation combining two dynamics. Let $k_1 \in L_{1,loc}(\mathbb{R}_+)$ and $k_2 \in L_{1,loc}(\mathbb{R}_+)$ and set

$$d\mu = \mu_0 k_1(t)dt, \quad d\varepsilon = \varepsilon_0 k_2(t)dt, \quad d\overline{\sigma}_R = \frac{1}{\sigma_0} d\delta_0, \quad d\sigma_G \equiv 0,$$

in (3.15), where μ_0, ε_0 and σ_0 are positive constants. Then the general non-local telegraph equation takes the form

$$\partial_t^2 (k_1 * k_2 * u(\cdot,x))(t) + \eta \partial_t (k_2 * u(\cdot,x))(t) - \nu \partial_x^2 u(t,x) = 0, \quad t > 0, x \in \mathbb{R}, \tag{3.20}$$

where ν and η are defined as in the equation (1.10).

In this case, we note that the measures $d\mu$ and $d\varepsilon$ are induced by different functions k_1 and k_2, respectively. Therefore, it is clear that (3.20) depends on two functions k_1 and k_2. To due this, we say that (3.20) is non-local telegraph equation combining two dynamics. In the first instance it is not clear whether it is the dynamic of k_1 or of k_2 the governing dynamic of (3.20). Moreover, since this model has never been studied as a telegraph model, we do not know yet what is the relation with stochastic processes, but it is considered as a future work.

(5) The telegraph equation combining three dynamics. Let $k_1 \in L_{1,loc}(\mathbb{R}_+)$, $k_2 \in L_{1,loc}(\mathbb{R}_+)$ and $k_3 \in L_{1,loc}(\mathbb{R}_+)$ and set

$$d\mu = \mu_0 k_1(t)dt, \quad d\varepsilon = \varepsilon_0 k_2(t)dt, \quad d\overline{\sigma}_R = \frac{1}{\sigma_0} k_3(t)dt, \quad d\sigma_G \equiv 0,$$

in (3.15), where μ_0, ε_0 and σ_0 are positive constants. Then the general non-local telegraph equation takes the form

$$\partial_t^2(k_1 * k_2 * u(\cdot, x))(t) + \eta\partial_t(k_2 * k_3 * u(\cdot, x))(t) - \nu\partial_x^2 u(t, x) = 0, \quad t > 0, x \in \mathbb{R}, \tag{3.21}$$

where ν and η are defined as in the equation (1.10).

***Remark* 3.1.** *All models not considered in this thesis are topics considered for future research.*

Chapter 4

Non-local in time telegraph equation and very slowly growing variance

This Chapter is based on the article ***"Non-local in time telegraph equation and very slowly growing variance"***, see [4], in collaboration with Juan Carlos Pozo (Universidad de Chile).

In this Chapter, we want to study the asymptotic behavior of the variance of the stochastic process associated with the fundamental solution of the equation (1.10) at large and short times. In this context, we developed a method to construct new examples of kernels such that its variance has a slow growth, extending the result by Pozo and Vergara in [94], which is the basis of this chapter. Also, we show that our approach can be adapted to sub-diffusion process.

4.1 Very slowly growing variance.

We start by recalling the result established in [94, Theorem 1.1].

Theorem 4.1. *Let η, ν be positive constants and $(k, \ell) \in (\mathcal{PC})$. The variance* $\mathrm{Var}[X(t)]$ *of the process $X(t)$, whose density function coincides with the fundamental solution of* (1.10)*, satisfies the following Volterra equation*

$$Var[X(t)] + 2\eta(\ell * Var[X(\cdot)])(t) = 2\nu^2(1 * \ell * \ell)(t), \quad t \geq 0. \tag{4.1}$$

Further, $Var[X(t)]$ is positive and increasing on $(0, \infty)$ and it satisfies the formula

$$Var[X(t)] = 2\nu^2(1 * \ell * r_{2\eta})(t), \quad t \geq 0. \tag{4.2}$$

Since (4.2) is given by means of convolutions, we can use Theorem 2.3, Karamata-Feller, to study the behavior of $\mathrm{Var}[X(t)]$ at large times and short times. The following theorem is the main result of this section.

Theorem 4.2. *Let $(k, \ell) \in (\mathcal{PC})$. If the Laplace transform $\widehat{\ell}$ is a regularly varying function of index $\varrho_1 < \frac{1}{2}$, then*

$$\mathrm{Var}[X_k(t)] \sim \frac{2\nu^2}{\Gamma(1 - 2\varrho_1)} \left(\widehat{\ell}(t^{-1})\right)^2, \text{ as } t \to 0^+. \tag{4.3}$$

Further, if the function $t \mapsto \widehat{\ell}(t^{-1})$ is a regularly varying function of index $\varrho_2 > -1$, then

$$\mathrm{Var}[X_k(t)] \sim \frac{\nu^2}{\eta\,\Gamma(1 + \varrho_2)} \widehat{\ell}\Big(\frac{1}{t}\Big), \quad \text{as} \quad t \to \infty. \tag{4.4}$$

Proof. Set $V(t) = \mathrm{Var}[X(t)]$ for $t \geq 0$. It follows from (4.2) that the Laplace transform of V is given by

$$\widehat{V}(\lambda) = \frac{2\nu^2}{\lambda} \cdot \frac{\widehat{\ell}(\lambda)}{1 + 2\eta\widehat{\ell}(\lambda)} \cdot \widehat{\ell}(\lambda), \quad \lambda > 0.$$

Since $\widehat{\ell}(\lambda) \to 0$ as $\lambda \to \infty$, we have that

$$\widehat{V}(\lambda) \sim \frac{2\nu^2}{\lambda^{1-2\varrho_1}} L_1(\lambda), \quad \text{as} \quad \lambda \to \infty,$$

where $L_1(t) = t^{-2\varrho_1}\left(\widehat{\ell}(t)\right)^2$. Since $\widehat{\ell}$ is a regularly varying function of index ϱ_1, by Remark 2.3, we have that L_1 is a slowly varying function. Since $\frac{1}{2} > \varrho_1$, it follows from Theorem 2.3 that

$$\mathrm{Var}[X_k(t)] \sim \frac{2\nu^2}{\Gamma(1-2\varrho_1)} \left(\widehat{\ell}(t^{-1})\right)^2, \text{ as } t \to 0^+.$$

On the other hand, we note that

$$\widehat{V}(\lambda) = \frac{2\nu^2}{\lambda^{1+\varrho_2}} L_2\left(\frac{1}{\lambda}\right), \quad \lambda > 0,$$

where $L_2\colon (0,\infty) \to (0,\infty)$ is defined by

$$L_2(t) = \frac{\widehat{\ell}(t^{-1})}{1+2\eta\widehat{\ell}(t^{-1})} \cdot \frac{\widehat{\ell}(t^{-1})}{t^{\varrho_2}}, \quad \text{for} \quad t > 0.$$

Since $t \mapsto \widehat{\ell}(t^{-1})$ is a regularly varying function of index ϱ_2, by Remark 2.3 we have that L_2 is a slowly varying function. Furthermore,

$$L_2(\lambda^{-1}) \sim \frac{1}{2\eta\lambda^{\varrho_2}} \widehat{\ell}(\lambda^{-1}) \quad \text{as} \quad \lambda \to 0^+.$$

Since $\varrho_2 > -1$, it follows from Theorem 2.3 that

$$\mathrm{Var}[X(t)] \sim \frac{\nu^2}{\eta} \cdot \frac{t^{\varrho_2}}{\Gamma(1+\varrho_2)} L_2(t) \sim \frac{\nu^2}{\eta\,\Gamma(1+\varrho_2)}\, \widehat{\ell}\left(\frac{1}{t}\right), \quad \text{as} \quad t \to \infty.$$

□

We remark that the conditions of Theorem 4.2 are satisfied for many pairs of functions $(k,\ell) \in (\mathcal{PC})$. For instance, all the examples presented in [94, Section 5] satisfy these conditions.

Example 4.1. *(Time fractional telegraph equation). If $k = g_{1-\alpha}$ with $\alpha \in (0,1)$, then*

$$Var[X_k(t)] \sim \frac{\nu^2}{\eta} \frac{t^\alpha}{\Gamma(1+\alpha)}, \quad as \quad t \to \infty,$$

and

$$Var[X_k(t)] \sim 2\nu^2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \quad as \quad t \to 0^+.$$

Example 4.2. *(Sum of two time fractional derivatives). If $k = g_{1-\alpha} + g_{1-\beta}$ with $0 < \alpha < \beta < 1$, then*

$$Var[X_k(t)] \sim \frac{\nu^2}{\eta} \frac{t^\alpha}{\Gamma(1+\alpha)}, \quad as \quad t \to \infty,$$

and

$$Var[X_k(t)] \sim 2\nu^2 \frac{t^{2\beta}}{\Gamma(1+2\beta)}, \quad as \quad t \to 0^+.$$

Example 4.3. *(Time fractional telegraph equation with Mittag-Leffler weight). If $k(t) = t^{\beta-1}E_{\alpha,\beta}(-\omega t^\alpha)$ with $0 < \alpha, \beta < 1$ and $\omega > 0$, then*

$$Var[X_k(t)] \sim \frac{\nu^2}{\eta} \frac{\omega\, t^{\alpha+1-\beta}}{\Gamma(2+\alpha-\beta)}, \quad as \quad t \to \infty,$$

and

$$Var[X_k(t)] \sim 2\nu^2 \frac{t^{2-2\beta}}{\Gamma(2+\alpha-\beta)}, \quad as \quad t \to 0^+.$$

Example 4.4. *(Time distributed order telegraph equation). If* $k(t) = \int_a^b g_\alpha(t)d\alpha$*, with* $0 \leq a < b \leq 1$*, then*

$$Var[X_k(t)] \sim \frac{\nu^2}{\eta}\frac{t^{1-b}\log(t)}{\Gamma(2-b)}, \quad as \quad t \to \infty,$$

and

$$Var[X_k(t)] \sim 2\nu^2\,\frac{t^{1+b-2a}\big(\log(t)\big)^2}{\Gamma(2-b)}, \quad as \quad t \to 0^+.$$

***Remark* 4.1.** *We note that if* $b = 1$ *and* $a \in (0,1)$ *in Example 4.4 then we have an infinity family of processes whose variance behaves like a logarithmic function at infinity.*

The following example was not available in the literature before. De define recursively the functions Θ_n as follows

$$\Theta_1(t,x) = \int_0^x g_\alpha(t)d\alpha \quad \text{and} \quad \Theta_{n+1}(t,x) = \int_0^x \Theta_n(t,y)dy, \quad t > 0,\ x > 0. \tag{4.5}$$

Further, for $n \in \mathbb{N}$ we define the functions $\theta_n \colon (0,\infty) \to (0,\infty)$ by

$$\theta_n(t) = \Theta_n(t,1), \text{ for } t > 0. \tag{4.6}$$

Example 4.5. *Let* $n \in \mathbb{N}$*. Consider* $k = \theta_n$ *where* θ_n *has been defined in* (4.6)*, then*

$$Var[X_k(t)] \sim \frac{\nu^2}{\eta}\big(\log(t)\big)^n, \quad as \quad t \to \infty,$$

and

$$Var[X_k(t)] \sim \nu^2\big((n-1)!\big)^2\big(t\cdot\log(t)\big)^2, \quad as \quad t \to 0^+.$$

Proof. Let $n \in \mathbb{N}$. We note that θ_n is positive and locally integrable on $\mathbb{R}_+$. Since $\alpha \in (0,1)$ we have that $g_\alpha \in (\mathcal{CM})$. We recall that the class of completely monotonic functions is closed under addition and pointwise limits, see [103, Corollary 1.6 and Corollary 1.7]. Therefore, we have that $\theta_n \in (\mathcal{CM})$ as well. It follows from [39, Theorem 5.4 and Theorem 5.5] that there exists $\zeta_n \in (\mathcal{CM})$ such that

$$\theta_n * \zeta_n = 1. \tag{4.7}$$

Consequently, we have that $(\theta_n, \zeta_n) \in (\mathcal{PC})$. Further, for $z > 0$ and $x > 0$ we note that $\widehat{\Theta}_1(z,x) = \dfrac{z^x - 1}{z^x\log(z)}$. Hence, it follows from (4.5) that

$$\widehat{\Theta}_n(z,x) = \frac{1}{z^x\big(\log(z)\big)^n}\left((-1)^n + z^x\sum_{k=1}^n \frac{(-1)^{k-1}x^{n-k}(\log(z))^{n-k}}{(n-k)!}\right), \ z > 0,\ x > 0.$$

This in turn implies that

$$\widehat{\theta}_n(\lambda) = \frac{1}{\lambda\big(\log(\lambda)\big)^n}\left((-1)^n + \lambda\sum_{k=1}^n \frac{(-1)^{k-1}(\log(\lambda))^{n-k}}{(n-k)!}\right), \quad \lambda > 0,$$

and

$$\widehat{\zeta}_n(\lambda) = \frac{\big(\log(\lambda)\big)^n}{(-1)^n + \lambda\displaystyle\sum_{k=1}^n \frac{(-1)^{k-1}(\log(\lambda))^{n-k}}{(n-k)!}}, \quad \lambda > 0.$$

By properties of logarithmic functions, we have that $\widehat{\zeta}_n(t)$ is a regularly varying function of index $\varrho = -1$ and $\widehat{\zeta}_n(t^{-1})$ is a slowly varying function. Therefore, by a direct calculation and using Theorem 4.2, we have that

$$\text{Var}[X_k(t)] \sim \frac{\nu^2}{\eta}\big(\log(t)\big)^n, \quad \text{as} \quad t \to \infty,$$

and

$$\text{Var}[X_k(t)] \sim \nu^2\big((n-1)!\big)^2\big(\big(t\cdot\log(t)\big)^2, \quad \text{as} \quad t \to 0^+.$$

□

We have mentioned at the beginning of this section, we are interested into find examples of $(k,\ell)\in(\mathcal{PC})$ such that $\mathrm{Var}[X(t)]$ grows slower than a logarithmic function at infinity. To this end, we prove the following result.

Lemma 4.1. *Let $f,g\in L_{1,loc}(\mathbb{R}_+)$. Assume that $f,g\in(\mathcal{CM})$, then there exists $h\in(\mathcal{CM})$ such that*

$$\widehat{h}(z)=\widehat{f}(z)\ \widehat{g}(\widehat{f}(z)),\quad \lambda>0.$$

Proof. Consider $a=1*f$, $b\equiv 1$, $c=1*g$. It is clear that a,b and c are Bernstein functions. According to Lemma 2.2, there exists a Bernstein function $e\colon(0,\infty)\to(0,\infty)$ such that $e(0^+)=0$ and

$$\widehat{e}(\lambda)=\widehat{a}(\lambda)\widehat{dc}\left(\frac{\widehat{a}(\lambda)}{\widehat{b}(\lambda)}\right),\ \lambda>0.$$

In consequence, defining $h(t)=\frac{d}{dt}e(t)$ we have that $h\in(\mathcal{CM})$ and

$$\widehat{h}(\lambda)=\lambda\widehat{e}(\lambda)-e(0^+)=\widehat{f}(\lambda)\ \widehat{g}(\widehat{f}(\lambda)),\quad \lambda>0.$$

□

Corollary 4.1. *For all $\delta\in(0,1]$ there exists a pair $(\phi_1^\delta,\psi_1^\delta)\in(\mathcal{PC})$ such that*

$$\widehat{\psi_1^\delta}(t^{-1})\sim\big(\log(t)\big)^\delta,\quad as\quad t\to\infty,$$

and

$$\widehat{\psi_1^\delta}(t^{-1})\sim\big(t\cdot\log(t^{-1})\big)^\delta\quad as\quad t\to 0^+.$$

If $\delta=1$, we will simply write $(\phi_1,\psi_1)\in(\mathcal{PC})$.

Proof. Let $\delta\in(0,1]$. Consider the pair $(\theta,\zeta)\in(\mathcal{PC})$ given by

$$\theta(t)=\int_0^1\frac{t^{\alpha-1}}{\Gamma(\alpha)}d\alpha,\quad\text{and}\quad \zeta(t)=\int_0^\infty\frac{e^{-st}}{1+s}ds,\quad t>0. \tag{4.8}$$

It is a well-known fact that both θ and ζ are completely monotonic functions. So, applying Lemma 4.1 with $f=\theta$ and $g=g_{1-\delta}$, we conclude that there exists $h_{1\delta}\in(\mathcal{CM})$ such that

$$\widehat{h_{1\delta}}(\lambda)=\left(\frac{\lambda-1}{\lambda\ \log(\lambda)}\right)^\delta,\quad \lambda>0.$$

Applying again Lemma 4.1 with $f=\zeta$ and $g=g_{1-\delta}$, we note that there exists $h_{2\delta}\in(\mathcal{CM})$ such that

$$\widehat{h_{2\delta}}(\lambda)=\left(\frac{\log(\lambda)}{\lambda-1}\right)^\delta,\quad \lambda>0.$$

Now define the kernels

$$\phi_1^\delta=g_{1-\delta}*h_{2\delta},\quad\text{and}\quad \psi_1^\delta=h_{1\delta}.$$

Applying directly the Laplace transform, we have

$$\widehat{\phi_1^\delta}(\lambda)=\frac{1}{\lambda}\left(\frac{\lambda-1}{\log(\lambda)}\right)^\delta,\quad\text{and}\quad \widehat{\psi_1^\delta}(\lambda)=\left(\frac{\log(\lambda)}{\lambda-1}\right)^\delta,\quad \lambda>0. \tag{4.9}$$

We note that by construction $\psi_1^\delta\in(\mathcal{CM})$. Hence, it follows from [39, Theorem 5.4 and Theorem 5.5] that $\phi_1^\delta\in(\mathcal{CM})$. In consequence $(\phi_1^\delta,\psi_1^\delta)\in(\mathcal{PC})$. Furthermore, it clear that

$$\widehat{\psi_1^\delta}(t^{-1})=\left(\frac{\log(t)}{1-t^{-1}}\right)^\delta,\quad t>0,$$

which in turn implies that

$$\widehat{\psi_1^\delta}(t^{-1}) \sim \big(\log(t)\big)^\delta, \quad \text{as} \quad t \to \infty.$$

To compute the behavior $\widehat{\psi_1^\delta}(t^{-1})$ as $t \to 0^+$, we rewrite this function as follows

$$\widehat{\psi}_\delta(t^{-1}) = \left(\frac{t\,\log(t)}{t-1}\right)^\delta, \quad t > 0,$$

which implies that

$$\widehat{\psi}_\delta(t^{-1}) \sim \big(-t \cdot \log(t)\big)^\delta \quad \text{as} \quad t \to 0^+.$$

□

***Remark* 4.2.** *Let $\delta \in (0,1]$ and $(\phi_\delta, \psi_\delta) \in (\mathcal{PC})$ given in Corollary 4.1. If $\delta = 1$, then (ϕ_1, ψ_1) is the same pair of functions defined by Kochubei in [60]. On the other hand, since $\widehat{\psi_1^\delta}(t^{-1}) \sim (\log(t))^\delta$ as $t \to \infty$ we have that $\psi_1^\delta \notin L_1(\mathbb{R}_+)$.*

Example 4.6. *Let $\delta \in (0,1]$. Consider the pair $(k, \ell) = (\phi_1^\delta, \psi_1^\delta)$ given in Corollary 4.1, then*

$$Var[X_k(t)] \sim \frac{\nu^2}{\eta}\big(\log(t)\big)^\delta, \quad as \quad t \to \infty,$$

and

$$Var[X_k(t)] \sim 2\nu^2 \frac{\big(t \cdot \log(t^{-1})\big)^{2\delta}}{\Gamma(1+2\delta)}, \quad as \quad t \to 0^+.$$

Proof. It follows from (4.9) that $\widehat{\ell}$ is a regularly varying function of index $\varrho = -\delta$. Further, since

$$\widehat{\ell}(t^{-1}) \sim (\log(t))^\delta, \text{ as } t \to \infty,$$

it follows that $\widehat{\ell}(t^{-1})$ is a slowly varying function. Therefore, Theorem 4.2 implies that

$$\text{Var}[X_k(t)] \sim \frac{\nu^2}{\eta}\big(\log(t)\big)^\delta, \quad \text{as} \quad t \to \infty,$$

and

$$\text{Var}[X_k(t)] \sim 2\nu^2 \frac{\big(t \cdot \log(t^{-1})\big)^{2\delta}}{\Gamma(1+2\delta)}, \quad \text{as} \quad t \to 0^+.$$

□

Corollary 4.2. *For all $n \in \mathbb{N}$ there exists a pair $(\phi_n, \psi_n) \in (\mathcal{PC})$ such that $\phi_n \in (\mathcal{CM})$ and*

$$\widehat{\psi_n}(t^{-1}) \sim \log^{[n]}(t), \; as \; t \to \infty,$$

and

$$\widehat{\psi_n}(t^{-1}) \sim t\,\big(\log(t^{-1})\big)^n \; as \; t \to 0^+,$$

where $\log^{[n]} = \underbrace{\log \circ \log \circ \cdots \circ \log}_{n-\text{times}}$.

Proof. The proof will be done by induction. For $n = 1$, we consider the pair $(\phi_1, \psi_1) \in (\mathcal{PC})$ given by Corollary 4.1 with $\delta = 1$.
Let $n > 1$. Assume that there exists a pair $(\phi_n, \psi_n) \in (\mathcal{PC})$ such that $\phi_n \in (\mathcal{CM})$ satisfying

$$\widehat{\psi}_n(t^{-1}) \sim \log^{[n]}(t), \; \text{as } t \to \infty.$$

and

$$\widehat{\psi_n}(t^{-1}) \sim t\,\big(\log(t^{-1})\big)^n, \; \text{as } t \to 0^+.$$

Since $\phi_n \in (\mathcal{CM})$ and $\phi_n * \psi_n = 1$, it follows from Lemma 4.1 that for all $\delta \in (0,1)$ there exists a completely monotonic function, denoted by φ_n^δ, such that

$$\widehat{\varphi_n^\delta}(\lambda) = \frac{1}{\lambda}\big(\widehat{\psi}_n(\lambda)\big)^{-\delta}, \quad \lambda > 0.$$

Now define

$$\phi_{n+1}(t) = \int_0^1 \varphi_n^\delta(t)d\delta, \quad t > 0. \tag{4.10}$$

Since the class of completely monotonic functions is closed under addition and pointwise limits, we conclude that $\phi_{n+1} \in (\mathcal{CM})$. In consequence, it follows from [39, Theorem 5.4 and Theorem 5.5] that there exists $\psi_{n+1} \in (\mathcal{CM})$ such that

$$\psi_{n+1} * \varphi_{n+1} = 1. \tag{4.11}$$

Furthermore, we have that

$$\widehat{\phi_{n+1}}(\lambda) = \int_0^1 \widehat{\varphi_n^\delta}(\lambda)d\delta = \int_0^1 \frac{1}{\lambda}\Big(\widehat{\psi}_n(\lambda)\Big)^{-\delta} d\delta = \frac{\widehat{\psi}_n(\lambda) - 1}{\lambda\,\widehat{\psi}_n(\lambda)\,\log\big(\widehat{\psi}_n(\lambda)\big)}, \quad \lambda > 0.$$

Since $(\phi_{n+1}, \psi_{n+1}) \in (\mathcal{PC})$, this in turn implies that

$$\widehat{\psi_{n+1}}(\lambda) = \frac{\widehat{\psi}_n(\lambda)\log\big(\widehat{\psi}_n(\lambda)\big)}{\widehat{\psi}_n(\lambda) - 1}, \quad \lambda > 0. \tag{4.12}$$

Therefore, we have that

$$\widehat{\psi_{n+1}}(t^{-1}) = \frac{\widehat{\psi_n}(t^{-1})\log\big(\widehat{\psi_n}(t^{-1})\big)}{\widehat{\psi_n}(t^{-1}) - 1}, \quad t > 0.$$

We note that the inductive hypothesis implies that

$$\frac{\widehat{\psi}_n(t^{-1})}{\widehat{\psi}_n(t^{-1}) - 1} \sim \frac{\log^{[n]}(t)}{\log^{[n]}(t) - 1} \sim 1, \text{ as } t \to \infty.$$

In consequence we have

$$\widehat{\psi_{n+1}}(t^{-1}) \sim \log\big(\widehat{\psi}_n(t^{-1})\big) \sim \log^{[n+1]}(t), \text{ as } t \to \infty.$$

On the other hand, since $\widehat{\psi_n}(t^{-1}) \to 0$ as $t \to 0^+$, we have that

$$\widehat{\psi_{n+1}}(t^{-1}) \sim \widehat{\psi}_n(t^{-1})\log\big(\widehat{\psi}_n(t^{-1})\big), \text{ as } \quad t \to 0^+,$$

which by the inductive hypothesis is equivalent to

$$\widehat{\psi_{n+1}}(t^{-1}) \sim \left(t\left(\log(t^{-1})\right)^n\right)\Big(\log(t) + \log(\log(t^{-1})^n)\Big), \quad \text{as } t \to 0^+.$$

We recall that

$$\lim_{t\to 0^+} \frac{\log(\log(t^{-1})^n)}{\log(t)} = 0,$$

for all $n \in \mathbb{N}$. Hence

$$\widehat{\psi_{n+1}}(t^{-1}) \sim t\left(\log(t^{-1})\right)^{n+1}, \text{ as } t \to 0^+,$$

and the proof is complete. □

***Remark* 4.3.** *Let $n \in \mathbb{N}$ and $(\phi_n, \psi_n) \in (\mathcal{PC})$ given in Corollary 4.2. Since $\widehat{\psi_n}(t^{-1}) \sim \log^{[n]}(t)$ as $t \to \infty$, we have that for all $n \in \mathbb{N}$ the functions $\psi_n \notin L_1(\mathbb{R}_+)$.*

Example 4.7. *Let $n \in \{2, 3, \cdots\}$. Consider the pair $(k, \ell) = (\phi_n, \psi_n)$ given in Corollary 4.2, then*

$$Var[X(t)] \sim \frac{\nu^2}{\eta} \log^{[n]}(t), \quad as \quad t \to \infty,$$

and

$$Var[X(t)] \sim \nu^2 t^2 \left(\log(t^{-1})\right)^{2n}, \quad as \quad t \to 0^+,$$

where $\log^{[n]} = \underbrace{\log \circ \log \circ \cdots \circ \log}_{n-\text{times}}$.

Proof. We note that for every $n \in \mathbb{N}$ the function $\widehat{\psi_n}$ is a regularly varying function of index $\varrho = -1$. Indeed, recall that $\widehat{\psi}_1$ is defined by

$$\widehat{\psi}_1(t) = \frac{\log(t)}{t-1}, \quad t > 0.$$

Hence, it clear that $\widehat{\psi}_1$ is a regularly varying function of index $\varrho = -1$. Assume now that $\widehat{\psi}_n$ is a regularly varying function of index $\varrho = -1$ for some positive integer $n \geq 2$. Since $\widehat{\psi_n}(t) \to 0$ as $t \to \infty$, we have that

$$t \mapsto \frac{\widehat{\psi}_n(t)}{\widehat{\psi}_n(t) - 1}, \ t > 0,$$

is a regularly varying function of index $\varrho = -1$. Moreover, by properties of the logarithmic function, we have that

$$t \mapsto \log(\widehat{\psi}_n(t)), \ t > 0,$$

is a slowly varying function. Therefore, it follows from (4.12) that $\widehat{\psi}_{n+1}$ is a regularly varying function of index $\varrho = -1$. On the other hand, it follows from Corollary 4.2 that

$$\widehat{\psi}_n(t^{-1}) \sim \log^{[n]}(t), \ \text{as } t \to \infty. \tag{4.13}$$

Therefore, $\widehat{\ell}(t^{-1})$ is a slowly varying function and Theorem 4.2 implies that

$$\text{Var}[X(t)] \sim \frac{\nu^2}{\eta} \log^{[n]}(t), \quad \text{as} \quad t \to \infty,$$

and

$$\text{Var}[X(t)] \sim \nu^2 t^2 \left(\log(t^{-1})\right)^{2n}, \quad \text{as} \quad t \to 0^+.$$

□

Example 4.8. *Let us going to construct the previous kernel for $n = 2$. For that, let us consider the pair $(\theta, \zeta) \in (\mathcal{PC})$ given by*

$$\theta(t) = \int_0^1 \frac{t^{\alpha-1}}{\Gamma(\alpha)} d\alpha, \quad and \quad \zeta(t) = \int_0^\infty \frac{e^{-st}}{1+s} ds, \quad t > 0.$$

Then we construct $\widehat{\psi_2}$ by applying the Lemma 4.1 as follows

$$\widehat{\psi_2}(\lambda) = \frac{\widehat{\zeta}(\lambda) \log\left(\widehat{\zeta}(\lambda)\right)}{\widehat{\zeta}(\lambda) - 1}.$$

Thus, we have that

$$\widehat{\psi_2}(\lambda) = \frac{\log\left(\dfrac{\log(\lambda)}{\lambda - 1}\right)}{1 - \dfrac{\lambda - 1}{\log(\lambda)}}.$$

It is easy to see that the function $t \mapsto \widehat{\psi_2}(t^{-1})$ is a slowly varying function and $\widehat{\psi_2}(t^{-1}) \sim \log(\log(t))$ as $t \to \infty$. Finally, according to Theorem 4.2 we have that

$$Var\left[X_k(t)\right] \sim \frac{\nu^2}{\eta} \log(\log(t)), \quad as\ t \to \infty.$$

Corollary 4.3. *For all $n \in \mathbb{N}$ and $\delta \in (0,1)$ there exists a pair $(\phi_n^\delta, \psi_n^\delta) \in (\mathcal{PC})$ such that $\phi_n^\delta \in (\mathcal{CM})$ and*

$$\widehat{\psi_n^\delta}(t^{-1}) \sim \left(\log^{[n]}(t)\right)^\delta, \text{ as } t \to \infty,$$

and

$$\widehat{\psi_n}(t^{-1}) \sim t^\delta \left(\log(t^{-1})\right)^{\delta n} \text{ as } t \to 0^+.$$

Proof. Let $n \in \mathbb{N}$ and $\delta \in (0,1)$. Consider the pair $(\phi_n, \psi_n) \in (\mathcal{PC})$ given in Corollary 4.2. According to Lemma 4.1 there are completely monotonic functions h_{1n}^δ and h_{2n}^δ such that

$$\widehat{h_{1n}^\delta}(\lambda) = \left(\widehat{\phi_n}(\lambda)\right)^\delta, \quad \lambda > 0,$$

and

$$\widehat{h_{2n}^\delta}(\lambda) = \left(\widehat{\psi_n}(\lambda)\right)^\delta, \quad \lambda > 0.$$

Now define the kernels

$$\phi_n^\delta = g_{1-\delta} * h_{2n}^\delta, \quad \text{and} \quad \psi_n^\delta = h_{1n}^\delta. \tag{4.14}$$

Applying directly the Laplace transform, we have

$$\widehat{\phi_n^\delta}(\lambda) = \frac{1}{\lambda}\left(\widehat{\phi_n}(\lambda)\right)^\delta \quad \text{and} \quad \widehat{\psi_n^\delta}(\lambda) = \left(\widehat{\psi_n}(\lambda)\right)^\delta, \quad \lambda > 0.$$

We note that by construction $\psi_n^\delta \in (\mathcal{CM})$. Hence, it follows from [39, Theorem 5.4 and Theorem 5.5] that $\phi_n^\delta \in (\mathcal{CM})$. In consequence $(\phi_n^\delta, \psi_n^\delta) \in (\mathcal{PC})$. The rest of the proof follows the same ideas of Corollary 4.2. ☐

Example 4.9. *Let $n \in \{2, 3, \cdots\}$ and $\delta \in (0,1)$. Consider the pair $(k, \ell) = (\phi_n, \psi_n)$ given in Corollary 4.3, then*

$$Var[X_k(t)] \sim \frac{\nu^2}{\eta}\left(\log^{[n]}(t)\right)^\delta, \quad as \quad t \to \infty,$$

and

$$Var[X_k(t)] \sim 2\nu^2 \frac{t^{2\delta}\left(\log(t^{-1})\right)^{2n\delta}}{\Gamma(1+2\delta)}, \quad as \quad t \to 0^+,$$

where $\log^{[n]} = \underbrace{\log \circ \log \circ \cdots \circ \log}_{n-\text{times}}$.

Proof. Following the same ideas of Corollary 4.2 we note that for all $n \in \mathbb{N}$ the functions $\widehat{\psi_n^\delta}$ are regularly varying functions of index $\varrho = -\delta$ and $\widehat{\psi_n^\delta}(t^{-1})$ is a slowly varying function. The rest of the proof is similar to the proof of Example 4.7. ☐

4.2 Application to ultra slow diffusion equations

There are another contexts where the pairs $(k, \ell) \in (\mathcal{PC})$ play a fundamental role. For example, this type of functions has been successfully exploited to study the so-called *sub-diffusion processes*, see [56, 93, 113] and references therein. In order to fix some ideas and explain why the results developed in this work could be interesting in the theory of subdiffusion processes, we consider the following equation

$$\partial_t(k * (u(\cdot, x) - u_0(x)))(t) - \Delta u(t,x) = 0, \quad t > 0, x \in \mathbb{R}^d, \tag{4.15}$$

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^d, \tag{4.16}$$

where k is a kernel of type $(\mathcal{PC})$ and u_0 is a given function. It has been proved in [56, Section 2] that the fundamental solution of (5.49) coincides with the probability density function of a stochastic process $X(t)$. Moreover, in [56, Lemma 2.1] it has been proved that the *mean square displacement* $M(t)$ of such process is given by

$$M(t) = 2d(1 * \ell)(t), \quad t \geq 0. \tag{4.17}$$

The function $M(t)$ allows to measure how fast or slow is the diffusion of the equation (5.49). It is worthwhile to mention that the slowest known rate of growth of $M(t)$ follows a logarithmic law, for instance see [60] and references therein. In such work Kochubei considered functions of the form

$$k(t) = \int_0^1 g_\alpha(t)\sigma(\alpha)d\alpha, \quad t > 0,$$

where $\sigma(t)$ with $t \in [0,1]$ is a continuous, non-negative function different from zero on a set of positive measure.

Those equations of the form (5.49) whose mean square displacement follows a logarithmic rate (or even slower) are known in the specialized literature as *ultra slow diffusion equations* and they are strongly related with *ultraslow inverse subordinators*, see [80].

Our work allows to study some ultra-slow diffusion equations which had not been considered before. For instance, the equation (5.49) with $(k,\ell) = (\phi_n, \psi_n)$ for some $n \in \mathbb{N}$, where (ϕ_n, ψ_n) has been defined in Corollary 4.2. According to (4.17) we have that the Laplace transform of M is given by

$$\widehat{M}(\lambda) = \frac{2d}{\lambda}\widehat{\psi_n}(\lambda), \quad \lambda > 0,$$

which by Karamata-Feller's Theorem 2.3 and the asymptotic behavior of $\widehat{\psi_n}$ given in (4.13) imply that

$$M(t) \sim 2d \log^{[n]}(t), \quad \text{as} \quad t \to \infty.$$

In consequence, the mean square displacement $M(t)$ grows (at infinity) slower than a logarithmic function.

This procedure can be applied to all the pairs of functions in $(\mathcal{PC})$ defined in the Corollary 4.1 and Corollary 4.3. As far we know, this implies that there are an infinite number of ultra-slow diffusion equations which have not been analyzed before.

Chapter 5

Non-local in time telegraph equation and telegraph processes with random time

The results obtained in this Chapter are based on the submitted article ***"Non-local in time telegraph equation and telegraph processes with random time"***, [3], in collaboration with Verónica Poblete (Universidad de Chile) and Juan Carlos Pozo (Universidad de Chile).

In this chapter we study the properties of a non-markovian version of the telegraph process, whose non-markovian character comes from a non-local in-time evolution equation that is satisfied by its probability density function. In the first part of the Chapter, using the theory of Volterra integral equations, we obtain an explicit formula for its moments, and we prove that the Carleman condition is satisfied. This shows that the distribution of the process is uniquely determined by its moments. We also obtain an explicit formula for the moment generating function. In addition, we prove that the distribution of this process coincides with the distribution of a process of the form $T(|\mathcal{W}_{2k}(t)|)$ where $T(t)$ is the classical telegraph process, and $|\mathcal{W}_{2k}(t)|$ is a random time whose distribution is related to a non-local in-time version of the wave equation. To this end, we prove that the probability density function can be constructed via subordination from the distribution of the classic telegraph process.

We recall that from [94, Section 4] we have that the fundamental solution of (1.10), U_k, satisfies

$$\widehat{U}_k(\lambda, x) = \frac{1}{2\lambda}\sqrt{\frac{\left(\lambda\widehat{k}(\lambda)\right)^2 + 2\eta\lambda\widehat{k}(\lambda)}{\nu^2}} \exp\left(-|x|\sqrt{\frac{\left(\lambda\widehat{k}(\lambda)\right)^2 + 2\eta\lambda\widehat{k}(\lambda)}{\nu^2}}\right), \quad \lambda > 0,\ x \in \mathbb{R}, \tag{5.1}$$

where $\widehat{U}_k$ is the Laplace transform of U_k. Then, by taking the Fourier transform to (5.1) we obtain

$$\widehat{\widehat{U}}_k(\lambda, \xi) = \frac{1}{\lambda}\frac{\left(\lambda\widehat{k}(\lambda)\right)^2 + 2\eta\lambda\widehat{k}(\lambda)}{\left(\lambda\widehat{k}(\lambda)\right)^2 + 2\eta\lambda\widehat{k}(\lambda) + \nu^2|\xi|^2}. \tag{5.2}$$

where $\widehat{\widehat{U}}_k$ represents the Fourier Laplace transform of the fundamental solution U_k.

5.1 Moments of the process $X_k(t)$

In order to learn more about the behavior of the stochastic process associated to the fundamental solution of equation (1.10), we want to extend the work done in [94] and [4], about the explicit form of its Variance and the asymptotic behavior of the same.

Let $n \in \mathbb{N}_0$. The n-th moment of the process $X_k(t)$ will be denoted by $M_n(t)$ and it is defined by

$$M_n(t) = \int_{\mathbb{R}} x^n U_k(t,x)dx, \quad t \geq 0. \tag{5.3}$$

Now, we find an explicit formula for $M_n(t)$ and we prove that they satisfy the so-called *Carleman condition*, see Theorem 1.1. Such condition implies that the distribution of the process $X_k(t)$ is uniquely determined by the moments. Thus, analogously to what happens with the telegraph process in the classical case, the Moment-Problem is completely solved for $X_k(t)$, see Section 1.2. Subsequently, we find an explicit formula for the moment generating function (see Theorem 5.11 below). Such a formula includes as particular case the formula of the moment generating function of the classical telegraph process.

Lemma 5.1. *Let $(k,\ell) \in (\mathcal{PC})$ and $n \in \mathbb{N}_0$. Then $M_n(t)$ is given by the formula*

$$M_n(t) = \begin{cases} 0, & \text{if } n = 2m+1, \text{ for some } m \in \mathbb{N}_0, \\ (2m)!\, \nu^{2m} \left(1 * \phi_\ell^{*(m)}\right)(t), & \text{if } n = 2m, \text{ for some } m \in \mathbb{N}, \end{cases} \tag{5.4}$$

where $\phi_\ell = r_{2\lambda} * \ell$, and $\phi_\ell^{*(m)}$ is defined recursively as follows

$$\phi_\ell^{*(m)} = \begin{cases} \phi_\ell, & m = 1, \\ \phi_\ell * \phi_\ell^{*(m-1)}, & m \geq 2. \end{cases}$$

First, let us look at the following calculation, which is simple but necessary to prove the previous lemma. Let f be the function $f(x) = \arctan(x)$. We note that the first two derivatives of f are given respectively by

$$f'(x) = \frac{1}{1+x^2}$$

and

$$f''(x) = \frac{2x}{(1+x^2)^2}$$

and they satisfy

$$(1+x^2)f'' + 2xf' = 0. \tag{5.5}$$

Applying the Leibnitz formula for $n-2$ derivatives, in both terms of (5.5), we obtain

$$\left((1+x^2)f''\right)^{(n-2)} = (1+x^2)(f'')^{(n-2)} + \binom{n-2}{1}(2x)(f'')^{(n-3)} + \binom{n-2}{2}2(f'')^{(n-4)}$$

and

$$(2xf')^{(n-2)} = -2x(f')^{(n-2)} + \binom{n-2}{1}2(f')^{(n-3)}.$$

Thus, we can rewrite (5.5) as

$$(1+x^2)f^{(n)} + 2x(n-2)f^{(n-1)} + (n-2)(n-3)f^{(n-2)} + 2xf^{(n-1)} + 2(n-2)f^{(n-2)} = 0. \tag{5.6}$$

Then, we can calculate the nth derivative of $\arctan(x)$ in $x_0 = 0$ with the following recurrence formula

$$f^{(n)}(0) = -(n-2)(n-1)f^{(n-2)}(0).$$

Finally, we have that $f^{(0)}(0) = 0$, $f^{(1)}(0) = 1$ and we can conclude that

$$f^{(n)}(0) = \begin{cases} 0, & \text{if } n = 2k, \quad k \in \mathbb{N}, \\ (-1)^k(2k)!, & \text{if } n = 2k+1, \quad k \in \mathbb{N}_0. \end{cases} \tag{5.7}$$

According to equation (5.7), a straightforward calculation shows that

$$\partial_x^{2m}\left(\frac{1}{a+bx^2}\right)\bigg|_{x=0} = \frac{(-1)^m(2m)!b^m}{a^{m+1}}. \tag{5.8}$$

Proof of Lemma 5.1. Let $n \in \mathbb{N}_0$ and consider M_n defined as (2.25). For the Laplace transform M_n we have

$$\begin{aligned}\widehat{M_n}(\lambda) &= \int_0^\infty \int_{\mathbb{R}} x^n e^{-\lambda t} U_k(t,x)dx\,dt \\ &= i^n \int_0^\infty \int_{\mathbb{R}} (\partial_\xi^n e^{-ix\xi})\big|_{\xi=0} e^{-\lambda t} U_k(t,x)dx\,dt \\ &= i^n \partial_\xi^n \Big(\int_0^\infty \int_{\mathbb{R}} e^{-ix\xi} e^{-\lambda t} U_k(t,x)dx\,dt \Big)\Big|_{\xi=0} = i^n \partial_\xi^n \Big(\widetilde{\widehat{U_k}}(\lambda,\xi)\Big)\Big|_{\xi=0}.\end{aligned}$$

It follows from Equation (5.2) that

$$\widetilde{\widehat{U_k}}(\lambda,\xi) = \frac{1}{\lambda}\frac{\mathcal{K}(\lambda)}{\mathcal{K}(\lambda)+\nu^2\xi^2}, \quad \lambda > 0,\ \xi \in \mathbb{R},$$

where $\mathcal{K}(\lambda) = (\lambda\widehat{k}(\lambda))^2 + 2\mu\,\lambda\widehat{k}(\lambda)$. Therefore, we have that

$$\widehat{M_n}(\lambda) = i^n \frac{\mathcal{K}(\lambda)}{\lambda}\,\partial_\xi^n \left(\frac{1}{\mathcal{K}(\lambda)+\nu^2\xi^2}\right)\Bigg|_{\xi=0}, \quad \lambda > 0.$$

According to Equation (5.8) we have that

$$\widehat{M_n}(\lambda) = \begin{cases} 0, & \text{if } n = 2m+1, \text{ for some } m \in \mathbb{N}_0, \\ \dfrac{1}{\lambda}\cdot\dfrac{(2m)!\,\nu^{2m}}{\big((\lambda\widehat{k}(\lambda))^2 + 2\eta\, z\widehat{k}(\lambda)\big)^m}, & \text{if } n = 2m, \text{ for some } m \in \mathbb{N}. \end{cases}$$

Now, we note that

$$\frac{1}{(\lambda\widehat{k}(\lambda))^2 + 2\eta\,\lambda\widehat{k}(\lambda)} = \frac{1}{\lambda\widehat{k}(\lambda)}\cdot\frac{1}{\lambda\widehat{k}(\lambda)+2\eta} = \widehat{\ell}(\lambda)\cdot\widehat{r}_{2\eta}(\lambda) = \widehat{\phi}_\ell(\lambda), \quad \lambda > 0.$$

Thus

$$\widehat{M_n}(\lambda) = \begin{cases} 0, & \text{if } n = 2m+1, \text{ for some } m \in \mathbb{N}_0, \\ \dfrac{1}{\lambda}\cdot(2m)!\nu^{2m}\big(\widehat{\phi}_\ell(\lambda)\big)^m, & \text{if } n = 2m, \text{ for some } m \in \mathbb{N}, \end{cases}$$

and the representation (5.4) follows from the uniqueness of the Laplace transform. □

5.2 Moment Problem

In this section we give the complete solution to the Hamburger Moment-Problem for the stochastic process $X_k(t)$ associated to the fundamental solution of the equation (1.10), see (1.13). We show that, for any fixed $t > 0$, the moments of $X_k(t)$ satisfy the Carleman condition, see Theorem 1.1. Therefore, the distribution of $X_k(t)$ is completely determined by its moments. This result is given by the following Theorem.

Theorem 5.1. *For any fixed $t > 0$, the moments of $X_k(t)$ satisfy the Carleman condition*

$$\sum_{m=1}^{\infty} \frac{1}{\sqrt[2m]{M_{2m}(t)}} = \infty.$$

Proof. Let $t > 0$ be fixed. The condition $(k,\ell) \in (\mathcal{PC})$ implies that both $r_{2\mu}(t)$ and $\ell(t)$ are non-negative functions. Therefore, it follows directly from (2.7) that $r_{2\mu}(t) \le \ell(t)$, and consequently

$$M_{2m}(t) \le (2m)!\,\nu^{2m}(1 * \ell^{(*2m)})(t). \tag{5.9}$$

Further, using $2m$-times the Young inequality, we have that

$$M_{2m}(t) \le (2m)!\,\nu^{2m}\big((1*\ell)(t)\big)^{2m}.$$

This implies that

$$\frac{1}{\nu^2(1*\ell)(t)\sqrt[2m]{(2m)!}} \leq \frac{1}{\sqrt[2m]{M_{2m}(t)}},$$

and

$$\frac{1}{\nu^2(1*\ell)(t)}\sum_{m=1}^{\infty}\frac{1}{\sqrt[2m]{(2m)!}} \leq \sum_{m=1}^{\infty}\frac{1}{\sqrt[2m]{M_{2m}(t)}}.$$

It follows from Stirling formula that

$$\lim_{m\to\infty}\frac{m}{\sqrt[2m]{(2m)!}} = \frac{e}{2}.$$

Therefore, by the limit comparison test for divergent series, we have that

$$\frac{1}{\nu^2(1*\ell)(t)}\sum_{m=1}^{\infty}\frac{1}{\sqrt[2m]{(2m)!}}$$

is a divergent series, which implies that the series

$$\sum_{m=1}^{\infty}\frac{1}{\sqrt[2m]{M_{2m}(t)}},$$

is divergent as well. □

***Remark* 5.1.** *When $\ell \equiv 1$, it follows from (5.9) that*

$$M_{2m}(t) \leq \nu^{2m}\, t^{2m}, \quad t > 0. \tag{5.10}$$

This estimates improves the one obtained in [64, Theorem 3];

$$M_{2m}(t) \leq (\nu t)^{2m}(1+\mu t)^{\frac{\mu^2 t^2}{2}}, \quad t \geq 0.$$

In the case that $\ell = g_\alpha$ for some $\alpha \in (0,1)$, we have that

$$M_{2m}(t) \leq (2m)!\nu^{2m}\frac{t^{2m\alpha}}{\Gamma(2m\alpha+1)}, \quad t \geq 0.$$

Moment generating function

According to Equation (2.18), the moment generating function is given by

$$\mathbf{M}(t,z) = \sum_{n=0}^{\infty}\frac{z^n\, M_n(t)}{n!}, \quad t > 0,\ |z| < R_0, \tag{5.11}$$

for some $R_0 > 0$.

Once the distribution of a stochastic process is proved to be uniquely determined by the moments, one may wonder if a formula of the corresponding moment generating function can be obtained in an explicit form.

Theorem 5.2. *Let $(k,\ell) \in (\mathcal{PC})$. For any $t > 0$ and $|z| < R_0$, the moment generating function (5.11) has the form*

$$\mathbf{M}(t,z) = \frac{1}{2}\left[\left(\frac{\eta}{\sqrt{\eta^2+z^2\nu^2}}+1\right)s\big(t,Q_z^1\big) + \left(1-\frac{\eta}{\sqrt{\eta^2+z^2\nu^2}}\right)s\big(t,Q_z^2\big)\right], \tag{5.12}$$

where $Q_z^1 = \eta - \sqrt{\eta^2+\nu^2 z^2}$, $Q_z^2 = \eta + \sqrt{\eta^2+\nu^2 z^2}$ and $s(t,\cdot)$ is the scalar resolvent defined in (2.6).

Proof. According to (5.4) and (5.11), we have that

$$\mathbf{M}(t,z)=\sum_{n=0}^{\infty}\frac{z^{2n}\,M_{2n}(t)}{(2n)!},\quad t>0,\ |z|<R_0,$$

and its Laplace transform is given by

$$\widehat{\mathbf{M}}(\lambda,z)=\frac{1}{\lambda}\sum_{n=0}^{\infty}\left(\frac{\nu^2\,z^2\left(\widehat{\ell}(\lambda)\right)^2}{1+2\eta\widehat{\ell}(\lambda)}\right)^n,\quad \lambda>0,\ |z|<R_0,$$

Since $|z|<R_0$ and $\widehat{\ell}(\lambda)\to 0$ as $\lambda\to\infty$, there exists $\lambda_0>0$ large enough such that

$$\left|\frac{\nu^2\,z^2\left(\widehat{\ell}(\lambda)\right)^2}{1+2\eta\widehat{\ell}(\lambda)}\right|<1,$$

for all $\lambda>\lambda_0$. This implies that

$$\widehat{\mathbf{M}}(\lambda,z)=\frac{1}{\lambda}\frac{1}{1-\frac{\nu^2\,z^2\left(\widehat{\ell}(\lambda)\right)^2}{1+2\eta\widehat{\ell}(\lambda)}}\quad \lambda>\lambda_0,\ |z|<R_0.$$

Using the fact that $(k,\ell)\in(\mathcal{PC})$ we have that

$$\widehat{\mathbf{M}}(\lambda,z)=\frac{\lambda(\widehat{k}(\lambda))^2+2\eta\widehat{k}(\lambda)}{(\lambda\widehat{k}(\lambda))^2+2\eta\lambda\widehat{k}(\lambda)-\nu^2z^2}\quad \lambda>\lambda_0,\ |z|<R_0.$$

A straightforward calculation shows that

$$\widehat{\mathbf{M}}(\lambda,z)=\frac{a_1(z)\,\widehat{k}(\lambda)}{\lambda\widehat{k}(\lambda)+b_1(z)}+\frac{a_2(z)\,\widehat{k}(\lambda)}{\lambda\widehat{k}(\lambda)+b_2(z)}\quad \lambda>\lambda_0,\ |z|<R_0,$$

where

$$a_1(z)=\frac{\eta+\sqrt{\eta^2+z^2\nu^2}}{2\sqrt{\eta^2+z^2\nu^2}},\quad\text{and}\quad a_2(z)=\frac{\sqrt{\eta^2+z^2\nu^2}-\eta}{2\sqrt{\eta^2+z^2\nu^2}},$$

and

$$b_1(z)=\eta-\sqrt{\eta^2+z^2\nu^2},\quad\text{and}\quad b_2(z)=\eta+\sqrt{\eta^2+z^2\nu^2}.$$

Therefore,

$$\widehat{\mathbf{M}}(\lambda,z)=a_1(z)\widehat{s}\big(\lambda,b_1(z)\big)+a_2(z)\widehat{s}\big(\lambda,b_2(z)\big)\quad \lambda>\lambda_0,\ |z|<R_0.$$

Hence, by uniqueness of Laplace transform, we have that (5.11) is given by the formula (5.12). □

***Remark* 5.2. Classical case.** *Note that in the case $\ell\equiv 1$, we have that $s(t,\nu)=\exp(-t\nu)$. Therefore, we have that*

$$\begin{aligned}\boldsymbol{M}(t,z)=&\frac{e^{-\eta t}}{2}\left[\left(\frac{\eta}{\sqrt{\eta^2+z^2\nu^2}}+1\right)e^{t\sqrt{\eta^2+z^2\nu^2}}+\left(1-\frac{\eta}{\sqrt{\eta^2+z^2\nu^2}}\right)e^{-t\sqrt{\eta^2+z^2\nu^2}}\right],\\ =&e^{-\eta t}\left(\cosh\left(t\sqrt{\eta^2+z^2\nu^2}\right)+\frac{\eta}{\sqrt{\eta^2+z^2\nu^2}}\sinh\left(t\sqrt{\eta^2+z^2\nu^2}\right)\right),\end{aligned}$$

for $t>0$ and $|z|<R_0$. This formula coincides with [64, Theorem 4] for the classical telegraph process.

***Remark* 5.3. Time fractional case.** *Consider the pair $(k,\ell)=(g_{1-\alpha},g_\alpha)$. Then, we have that*

$$\begin{aligned}\boldsymbol{M}(t,z)=\frac{1}{2}\Bigg[&\left(\frac{\eta}{\sqrt{\eta^2+z^2\nu^2}}+1\right)E_{\alpha,1}\left[-\left(\eta-\sqrt{\eta^2-z^2\nu^2}\right)t^\alpha\right]+\\ &\left(1-\frac{\eta}{\sqrt{\eta^2+z^2\nu^2}}\right)\left[E_{\alpha,1}\left[-\left(\eta+\sqrt{\eta^2-z^2\nu^2}\right)t^\alpha\right]\right].\end{aligned}$$

This is a new example, not considered in the literature, as far as we know.

5.3 Asymptotic Behavior of the Moments

As a by-product of the representation of $M_n(t)$, we can give a precise description of their asymptotic behavior when the function ℓ satisfies an additional regularity condition. To this end, we apply a version of the Karamata-Feller Tauberian theorem, see Theorem 2.3.

The following theorem extend the results presented by Alegría and Pozo in [4, Theorem 3.2], which are presented in this thesis as well, see Theorem 4.2. The extension is understood in the following sense. The results in [4, Theorem 3.2] merely analyze the asymptotic behavior of the variance, which is the second moment of the process. Theorem 5.3 gives a precise description of the asymptotic behavior of all the even moments.

Theorem 5.3. *Let $(k, \ell) \in (\mathcal{PC})$ and $n \in \mathbb{N}$. If $\widehat{\ell} \in \mathcal{RV}_\infty^{\varrho_1}$ for some $\varrho_1 < \frac{1}{2n}$, then*

$$M_{2n}(t) \sim \frac{(2n)!\,\nu^{2n}}{\Gamma(1-2n\varrho_1)}\big(\widehat{\ell}(t^{-1})\big)^{2n}, \text{ as } t \to 0^+. \tag{5.13}$$

Further, if $\ell \notin L_1(\mathbb{R}_+)$ and $\widehat{\ell} \in \mathcal{RV}_0^{\varrho_2}$ for some $\varrho_2 > -\frac{1}{n}$, then

$$M_{2n}(t) \sim \frac{(2n)!}{\Gamma(1+n\varrho_2)}\left(\frac{\nu^2}{\eta}\right)^n\big(\widehat{\ell}(t^{-1})\big)^n, \text{ as } t \to \infty. \tag{5.14}$$

If $\ell \in L_1(\mathbb{R}_+)$, then

$$M_{2n}(t) \sim (2n)!\,\nu^{2n}\left(\frac{\|\ell\|_1^2}{1+\eta\|\ell\|_1}\right)^n, \text{ as } t \to \infty. \tag{5.15}$$

Proof. It follows from (2.9) and (5.4) that the Laplace transform of $M_{2n}(t)$ is given by

$$\widehat{M}_{2n}(\lambda) = \nu^{2n}(2n)!\,\frac{(\widehat{\ell}(\lambda))^{2n}}{\lambda(1+\eta\widehat{\ell}(\lambda))^n}, \quad \lambda > 0.$$

Since $\widehat{\ell}(\lambda) \to 0$ as $\lambda \to \infty$, we have that

$$\widehat{M}_{2n}(\lambda) \sim \frac{(2n)!\nu^{2n}}{\lambda^{1-2n\varrho_1}}L_1(\lambda), \quad \lambda > 0,$$

where $L_1(\lambda) = \lambda^{-2n\varrho_1}(\widehat{\ell}(\lambda))^2$ for $\lambda > 0$. Since $\ell \in \mathcal{RV}_\infty^{\varrho_1}$, it follows from Remark 2.2 that $L_1 \in \mathcal{SV}_\infty$. Moreover, since $\varrho_1 < \frac{1}{2n}$ it follows from Karamata-Feller Theorem 2.3 that

$$M_{2n}(t) \sim \frac{(2n)!\,\nu^{2n}}{\Gamma(1-2n\varrho_1)}\big(\widehat{\ell}(t^{-1})\big)^{2n}, \text{ as } t \to 0^+.$$

On the other hand, if $\ell \notin L_1(\mathbb{R}_+)$ we note that

$$\widehat{M}_{2n}(\lambda) \sim (2n)!\left(\frac{\nu^2}{\mu}\right)^n\frac{1}{\lambda^{1+n\varrho_2}}L_2\left(\frac{1}{\lambda}\right), \text{ as } \lambda \to 0,$$

where $L_2\colon (0,\infty) \to (0,\infty)$ is defined by $L_2(t) = \big(t^{-\varrho_2}\widehat{\ell}(t^{-1})\big)^n$, for $t > 0$. Since $\widehat{\ell} \in \mathcal{RV}_0^{\varrho_2}$, it follows from Remark 2.2 that $L_2 \in \mathcal{SV}_\infty$. Moreover, since $\varrho_2 > -\frac{1}{n}$, it follows from Karamata-Feller Theorem 2.3 that

$$M_{2n}(t) \sim \frac{(2n)!}{\Gamma(1+n\varrho_2)}\left(\frac{\nu^2}{\eta}\right)^n\big(\widehat{\ell}(t^{-1})\big)^n, \text{ as } t \to \infty.$$

To finish the proof, we note that if $\ell \in L_1(\mathbb{R}_+)$, then $\widehat{\ell}(\lambda) \to \|\ell\|_1$ as $\lambda \to 0$. Therefore, we have that

$$\widehat{M}_{2n}(\lambda) \sim \frac{1}{\lambda}\,(2n)!\,\nu^{2n}\left(\frac{\|\ell\|_1^2}{1+\eta\|\ell\|_1}\right)^n, \text{ as } \lambda \to 0.$$

Since the constant functions are slowly varying at infinity, we have that

$$M_{2n}(t) \sim (2n)!\,\nu^{2n}\left(\frac{\|\ell\|_1^2}{1+\eta\|\ell\|_1}\right)^n, \text{ as } t \to \infty.$$

□

We point out that the conditions of Theorem 5.3 might seem very restrictive. However, we will show that such conditions are satisfied by many pairs of functions $(k,\ell)\in(\mathcal{PC})$.

Example 5.1. *Consider the pair $(k,\ell)\in(\mathcal{PC})$ given by $(g_{1-\alpha},g_\alpha)$ for some $\alpha\in(0,1)$. It is well-known that*

$$\widehat{\ell}(\lambda)=\frac{1}{\lambda^\alpha},\qquad \lambda>0.$$

Therefore, we have that $\widehat{\ell}\in\mathcal{RV}_\infty^{\varrho_1}$ with $\varrho_1=-\alpha$, and $\widehat{\ell}\in\mathcal{RV}_0^{\varrho_2}$ with $\varrho_2=\alpha$. Since $\ell\notin L_1(\mathbb{R}_+)$, it follows from Theorem 5.3 that

$$M_{2n}(t)\sim\frac{(2n)!\,\nu^{2n}}{\Gamma(1+2n\alpha)}t^{2\alpha n},\ \text{as } t\to0^+,$$

and

$$M_{2n}(t)\sim\frac{(2n)!}{\Gamma(1+n\alpha)}\Big(\frac{\nu^2}{\eta}\Big)^n t^{\alpha n},\ \text{as } t\to\infty.$$

Example 5.2. *Consider the pair $(k_d,\ell_d)\in(\mathcal{PC})$ given by*

$$k_d(t)=\int_0^1 g_\alpha(t)d\alpha,\quad \text{and}\quad \ell_d(t)=\int_0^\infty\frac{e^{-st}}{1+s}ds,\quad t>0.$$

In this case, it has been proved in [60, Section 3] that $\ell_d\notin L_1(\mathbb{R}_+)$ and

$$\widehat{\ell}_d(\lambda)=\frac{\log(\lambda)}{\lambda-1},\qquad \lambda>0.$$

Therefore, we have that $\widehat{\ell}_d\in\mathcal{RV}_\infty^{\varrho_1}$ with $\varrho_1=-1$, and $\widehat{\ell}_d\in\mathcal{RV}_0^{\varrho_2}$ with $\varrho_2=0$. It follows from Theorem 5.3 that

$$M_{2n}(t)\sim\frac{(2n)!\,\nu^{2n}}{\Gamma(1+2n)}\big(t\log(t)\big)^{2n},\ \text{as } t\to0^+,$$

and

$$M_{2n}(t)\sim(2n)!\Big(\frac{\nu^2}{\eta}\Big)^n\big(\log(t)\big)^n,\ \text{as } t\to\infty.$$

Example 5.3. *Let $\delta\in(0,1)$ and consider the pair $(k_\delta,\ell_\delta)\in(\mathcal{PC})$ whose their Laplace transforms are given by*

$$\widehat{k}_\delta(\lambda)=\frac{1}{\lambda}\left(\frac{\lambda-1}{\log(\lambda)}\right)^\delta,\quad \text{and}\quad \widehat{\ell}_\delta(\lambda)=\left(\frac{\log(\lambda)}{\lambda-1}\right)^\delta,\quad \lambda>0.$$

This example was first considered in [4, Corollary 3.10]. In such a case, $\widehat{\ell}_\delta\in\mathcal{RV}_\infty^{\varrho_1}$ with $\varrho_1=-\delta$, and $\widehat{\ell}_\delta\in\mathcal{RV}_0^{\varrho_2}$ with $\varrho_2=0$. Thus, since $\ell_\delta\notin L_1(\mathbb{R}_+)$ it follows from Theorem 5.3 that

$$M_{2n}(t)\sim\frac{(2n)!\,\nu^{2n}}{\Gamma(1+2n)}\big(t\log(t)\big)^{2n\delta},\ \text{as } t\to0^+,$$

and

$$M_{2n}(t)\sim(2n)!\Big(\frac{\nu^2}{\eta}\Big)^n\big(\log(t)\big)^{n\delta},\ \text{as } t\to\infty.$$

Example 5.4. *Consider the pair $(k_{[2]},\ell_{[2]})\in(\mathcal{PC})$ whose their Laplace transforms are given by*

$$\widehat{k}_{[2]}(\lambda)=\frac{\log(\lambda)+1-\lambda}{\lambda\log(\lambda)\log\left(\frac{\log(\lambda)}{\lambda-1}\right)},\quad \text{and}\quad \widehat{\ell}_{[2]}(\lambda)=\frac{\log(\lambda)\log\left(\frac{\log(\lambda)}{\lambda-1}\right)}{\log(\lambda)+1-\lambda}\quad \lambda>0.$$

This example was first considered in [4, Corollary 3.13]. In such a case we have that $\widehat{\ell}_{[2]}\in\mathcal{RV}_\infty^{\varrho_1}$ with $\varrho_1=-1$, and $\widehat{\ell}_{[2]}\in\mathcal{RV}_0^{\varrho_2}$ with $\varrho_2=0$. It follows from Theorem 5.3 that

$$M_{2n}(t)\sim\frac{(2n)!\,\nu^{2n}}{\Gamma(1+2n)}\big(\log(t)\big)^{4n},\ \text{as } t\to0^+,$$

and

$$M_{2n}(t)\sim(2n)!\Big(\frac{\nu^2}{\eta}\Big)^n\big(\log(\log(t))\big)^n,\ \text{as } t\to\infty.$$

Example 5.5. *Let $\gamma > 0$, $\alpha \in (0,1)$ and consider the pair $(k,\ell) \in (\mathcal{PC})$ given by*

$$k(t) = g_{1-\alpha}(t)e^{-\gamma t} + \gamma \int_0^t g_{1-\alpha}(s)e^{-\gamma s}ds \quad \text{and} \quad \ell(t) = g_{\alpha,\gamma}(t) = g_\alpha(t)e^{-\gamma t}, \quad t > 0.$$

It is well-known that

$$\widehat{\ell}(\lambda) = \frac{1}{(\lambda+\gamma)^\alpha}, \quad \lambda > 0.$$

Therefore, we have that $\widehat{\ell} \in \mathcal{RV}_\infty^{\varrho_1}$ with $\varrho_1 = -\alpha$, and $\widehat{\ell} \in \mathcal{RV}_0^{\varrho_2}$ with $\varrho_2 = 0$. It follows from Theorem 5.3 that

$$M_{2n}(t) \sim \frac{(2n)!\,\nu^{2n}}{\Gamma(1+2n\alpha)} t^{2\alpha n}, \text{ as } t \to 0^+,$$

and

$$M_{2n}(t) \sim (2n)! \left(\frac{\nu^2}{\eta}\right)^n \gamma^{-\alpha n}, \text{ as } t \to \infty.$$

The preceding example is a particular case of the following more general situation.

Example 5.6. *Let $\gamma > 0$, $n \in \mathbb{N}$, $(k,\ell) \in (\mathcal{PC})$, $\widehat{\ell} \in \mathcal{RV}_\infty^{\varrho_1}$ with $\varrho_1 < \frac{1}{2n}$ and $\widehat{\ell} \in \mathcal{RV}_0^{\varrho_2}$. Consider the pair $(k_\gamma,\ell_\gamma) \in (\mathcal{PC})$ given by*

$$k_\gamma(t) = k(t)e^{-\gamma t} + \gamma \int_0^t k(s)e^{-\gamma s}ds \quad \text{and} \quad \ell_\gamma(t) = \ell(t)e^{-\gamma t}, \quad t > 0.$$

Note that, from the shift Laplace transform theorem, we have $\widehat{\ell}_\gamma(\lambda) = \widehat{\ell}(\lambda+\gamma)$. Since $\widehat{\ell} \in \mathcal{RV}_\infty^{\rho_1}$, it follows from [11, Proposition 1.5.7] that $\widehat{\ell}_\gamma \in \mathcal{RV}_\infty^{\rho_1}$. Moreover, since $\ell_\gamma \in L_1(\mathbb{R}^+)$, we have that $\widehat{\ell} \in \mathcal{RV}_0^{\varrho_2}$ with $\varrho_2 = 0$. Thus, it follows from Theorem 5.3 that

$$M_{2n}(t) \sim \frac{(2n)!\,\nu^{2n}}{\Gamma(1-2n\varrho_1)} \left(\widehat{\ell}_\gamma(t^{-1})\right)^{2n}, \text{ as } t \to 0^+,$$

and

$$M_{2n}(t) \sim (2n)! \left(\frac{\nu^2}{\eta}\right)^n \left(\widehat{\ell}(\gamma)\right)^n, \text{ as } t \to \infty.$$

Example 5.14 is interesting because it shows that there are infinitely many pairs $(k,\ell) \in (\mathcal{PC})$ such that $\ell \in L_1(\mathbb{R}_+)$ and the corresponding moments $M_{2n}(t)$ do not grow infinitely as time goes to infinity.

5.4 Subordination principle for the fundamental solution

We recall that Clement and Nohel [25, Theorem 2.1] established that the condition $(k,\ell) \in (\mathcal{PC})$ implies that ℓ is a *completely positive function*. This property has played a prominent role in the theory of Volterra equations. In particular, it has allowed to establish the *subordination principle in the sense of Prüss*, see [95, Chapter 4] , which has demonstrated to be a powerful tool in the theory of evolution equations. Using a similar approach, in this Section, we prove that it is possible to construct the fundamental solution of (1.10) from the fundamental solution of (1.4). In order to develop an iterative subordination method to this end, we extend the propagation function concept, see Definition 2.16.

If we define the function $\varphi_\sigma(\lambda) := \frac{1}{\sigma\widehat{\ell}(\lambda)}$, for $\sigma > 0$. It follows from [95, Proposition 4.5] that the condition $(\mathcal{PC})$ implies that φ_σ is a Bernstein function. This in turn implies that, for each fixed $\tau \geq 0$, the function

$$\psi_{\tau,\sigma}(\lambda) = \exp(-\tau\varphi_\sigma(\lambda)), \quad \lambda > 0,$$

is completely monotonic with respect to $\lambda > 0$. Therefore, from Bernstein's Theorem (see Theorem 2.1) for every $\tau \geq 0$, there is a unique nondecreasing function $W_{\ell,\sigma}(\cdot,\tau) \in BV(\mathbb{R}_+)$, normalized by $W_{\ell,\sigma}(0,\tau) = 0$ and left-continuous, such that

$$\widehat{W}_{\ell,\sigma}(\lambda,\tau) = \frac{\psi_{\tau,\sigma}(\lambda)}{\lambda}, \quad \lambda > 0. \tag{5.16}$$

The function $W_{\ell,\sigma}(\cdot,\cdot)$ is the propagation function, see Definition 2.16, associated to the completely positive function ℓ and the parameter σ. Note that, in the case $\sigma = 1$, this function coincides with the function (2.15), defined by Prüss in [95] and it has very interesting properties. The following result is a direct consequence of [95, Proposition 4.9].

Proposition 5.1. *Let $\sigma > 0$ and $(k,\ell) \in (\mathcal{PC})$. Then, for all $t \geq 0$, and $\nu \in \mathbb{R}_+$ we have that*

$$s(t,\sigma\nu) = -\int_0^\infty e^{-\tau\nu} d_\tau W_{\ell,\sigma}(t,\tau), \tag{5.17}$$

where $s(\cdot,\cdot)$ is the scalar resolvent function (see Definition 2.10 above).

Proof. Let $\sigma > 0$ and $\nu > 0$ and define the function

$$H(t) = -\int_0^\infty e^{-\tau\nu} d_\tau W_{\ell,\sigma}(t,\tau), \quad t > 0.$$

It follows from (5.16) that the Laplace transform of H is given by

$$\widehat{H}(\lambda) = \frac{\widehat{k}(\lambda)}{\sigma}\int_0^\infty e^{-\tau\nu} e^{-\frac{\tau\lambda\widehat{k}(\lambda)}{\sigma}} d\tau = \frac{\widehat{k}(\lambda)}{\sigma\nu + \lambda\widehat{k}(\lambda)}, \quad \lambda > 0.$$

On the other hand, it follows from (2.8) that

$$\widehat{s}(\lambda,\sigma\nu) = \frac{\widehat{k}(\lambda)}{\lambda\widehat{k}(\lambda) + \sigma\nu}, \quad \lambda > 0.$$

The conclusion follows from the uniqueness of Laplace transform. □

The preceding representation can be extended for complex numbers $\nu \in \mathbb{C}$ such that $Re(\nu) > 0$, provided that ℓ satisfies some additional assumptions such like being 1-regular, see Definition 2.14, and θ-sectorial, see Definition 2.13, as follow.

Proposition 5.2. *Let $\sigma > 0$ be fixed. Let $(k,\ell) \in (\mathcal{PC})$ be such that the function ℓ is 1-regular and θ-sectorial with $\theta < \frac{\pi}{2}$, then for every $t > 0$, we have that the propagation function $W_{\ell,\sigma}(t,\cdot) \in BV(\mathbb{R}_+) \cap W^{1,1}_{loc}(\mathbb{R}_+)$ and*

$$s(t,\sigma\nu) = -\int_0^\infty e^{-\nu\tau}\partial_\tau W_{\ell,\sigma}(t,\tau)d\tau, \quad t > 0,\ Re(\nu) > 0. \tag{5.18}$$

Proof. For $\sigma = 1$, this result has been established in [95, Proposition 3.5]. For any $\sigma > 0$, the proof follows from a rescaling argument of the case $\sigma = 1$. □

In the following result we prove that $\partial_\tau W_{\ell,\sigma}(t,\tau)$, for $t > 0$ and $\tau > 0$, can be interpreted as the probability density function of a stochastic process.

Proposition 5.3. *Let $\sigma > 0$ and $(k,\ell) \in (\mathcal{PC})$. If ℓ is 1-regular and θ-sectorial with $\theta < \frac{\pi}{2}$, then we have*

$$\partial_\tau(-W_{\ell,\sigma}(t,\tau)) = 2\,V_{2k}(t,\tau), \quad t > 0, \tau > 0,$$

where $W_{\ell,\sigma}(\cdot,\cdot)$ is the propagation associated to ℓ and σ, defined in (5.16), and $V_{2k}(\cdot,\cdot)$ denotes the solution of the following evolution equation

$$\partial_t^2(k * k * (v(\cdot,x) - \delta_0(x))(t) = \sigma^2\partial_x^2 v(t,x), \quad t > 0,\ x \in \mathbb{R}, \tag{5.19}$$

subject to the initial conditions $v(0,x) = \delta_0(x)$ and $\partial_t v(t,x)\big|_{t=0} = 0$, whenever the last one exists. In particular, $\partial_\tau W_{\ell,\sigma}(t,\tau) \geq 0$ for all $t \geq 0$ and $\tau > 0$, and

$$\int_0^\infty \partial_\tau(-W_{\ell,\sigma}(t,\tau))d\tau = 1,$$

for all $t \geq 0$.

Proof. Let $\sigma > 0$ be fixed. It follows from (5.16) that

$$-\partial_\tau \widehat{W}_{\ell,\sigma}(\lambda,\tau) = \frac{1}{\sigma\lambda\widehat{\ell}(\lambda)} \exp\Big(\frac{-\tau}{\sigma\widehat{\ell}(\lambda)}\Big), \quad \lambda > 0,\ \tau > 0.$$

From the condition $(k,\ell) \in (\mathcal{PC})$, we have that $\lambda\widehat{k}(\lambda)\widehat{\ell}(\lambda) = 1$ for all $\lambda > 0$, which in turn implies that

$$-\partial_\tau \widehat{W}_{\ell,\sigma}(\lambda,\tau) = \frac{\widehat{k}(\lambda)}{\sigma} \exp\Big(\frac{-\tau\lambda\widehat{k}(\lambda)}{\sigma}\Big), \quad \lambda > 0,\ \tau > 0.$$

On the other hand, according to (5.1) with $\eta = 0$, we have that the Laplace transform of the solution to (5.19) is given by

$$\widehat{V}_{2k}(\lambda,x) = \frac{1}{2\sigma}\widehat{k}(\lambda) \exp\Big(\frac{-|x|\lambda\widehat{k}(\lambda)}{\sigma}\Big), \quad \lambda > 0,\ x \in \mathbb{R}.$$

Thus, it follows from uniqueness of the Laplace transform that $\partial_\tau(-W_{\ell,\sigma}(t,\tau)) = 2\,V_{2k}(t,\tau)$, for $t > 0$ and $\tau > 0$. This fact shows that the function $-\partial_\tau W_{\ell,\sigma}(t,\tau)$ is positive for all $t \geq 0$ and $\tau > 0$. Moreover, it has been established

$$\int_{\mathbb{R}} V_{2k}(t,x)dx = 1,$$

for all $t \geq 0$. Since $V_{2k}(t,\cdot)$ is an even function, it follows that $\int_0^\infty \partial_\tau(-W_{\ell,\sigma}(t,\tau))d\tau = 1$, for all $t \geq 0$. □

***Remark* 5.4.** *We observe that Equation (5.19) can be obtained from Equation (1.10) by choosing $\eta = 0$ and $\nu = \sigma$. This fact implies that $V_{2k}(\cdot,\cdot)$ can be interpreted as the distribution of a stochastic process $\mathcal{W}(t)$, $t > 0$. Since $V_{2k}(t,\tau) = V_{2k}(t,-\tau)$ for all $t > 0$ and $\tau \in \mathbb{R}$, we can conclude that $2V_{2k}(t,\tau)$ with $t > 0$ and $\tau > 0$ is the probability density function of the process $|\mathcal{W}(t)|$, $t > 0$. In this sense we say that $V_{2k}(t,\tau)$ is the folded solution of (1.10).*

Now we are in position to prove that the fundamental solution of Equation (1.10) can be constructed via subordination from the distribution of the classical telegraph process.

Theorem 5.4. *Let $(k,\ell) \in (\mathcal{PC})$ such that ℓ is 1-regular and θ-sectorial with $\theta < \frac{\pi}{2}$. Then, the fundamental solution $U_k(t,x)$ of (1.10) can be represented as follows*

$$U_k(t,x) = -\int_0^\infty P(\tau,x)\partial_\tau W_{\ell,\sigma}(t,\tau)d\tau, \quad t > 0,\ x \in \mathbb{R}, \tag{5.20}$$

where $P(t,x)$ is the probability density function of the classical telegraph process, σ is an arbitrary positive constant, and $W_{\ell,\sigma}$ is the propagation function associated to the function ℓ and σ, defined in (5.16).

Proof. It follows from (5.2), that the Fourier-Laplace transform of U is given by

$$\widetilde{\widehat{U}}_k(\lambda,\xi) = \frac{1}{\lambda}\frac{\big(\lambda\widehat{k}(\lambda)\big)^2 + 2\eta\,\lambda\widehat{k}(\lambda)}{\big(\lambda\widehat{k}(\lambda)\big)^2 + 2\eta\,\lambda\widehat{k}(\lambda) + \nu^2\xi^2}, \quad \lambda > 0,\ \xi \in \mathbb{R}.$$

We decompose the denominator as follows

$$\big(\lambda\widehat{k}(\lambda)\big)^2 + 2\eta\,\lambda\widehat{k}(\lambda) + \nu^2\xi^2 = \big(\lambda\widehat{k}(\lambda) + \sigma R_1(\xi)\big)\cdot\big(\lambda\widehat{k}(\lambda) + \sigma R_2(\xi)\big),$$

where

$$R_1(\xi) = \frac{\eta - \sqrt{\eta^2 - \nu^2\xi^2}}{\sigma}, \quad \text{and} \quad R_2(\xi) = \frac{\eta + \sqrt{\eta^2 - \nu^2\xi^2}}{\sigma},$$

and σ is an arbitrary positive constant. By a straightforward computation we have that

$$\widetilde{\widehat{U}}_k(\lambda,\xi) = \left(\frac{1}{2} + \frac{\eta}{2\sqrt{\eta^2 - \nu^2\xi^2}}\right)\frac{\widehat{k}(\lambda)}{\lambda\widehat{k}(\lambda) + \sigma R_1(\xi)} + \left(\frac{1}{2} - \frac{\eta}{2\sqrt{\eta^2 - \nu^2\xi^2}}\right)\frac{\widehat{k}(\lambda)}{\lambda\widehat{k}(\lambda) + \sigma R_2(\xi)}, \quad \lambda > 0,\ \xi \in \mathbb{R}.$$

It follows from (2.8) that $\widetilde{U}(t,\xi)$, which is the Fourier transform of $U(t,x)$, is given by

$$\widetilde{U}_k(t,\xi) = \left(\frac{1}{2} + \frac{\eta}{2\sqrt{\eta^2 - \nu^2\xi^2}}\right) s(t,\sigma R_1(\xi)) + \left(\frac{1}{2} - \frac{\eta}{2\sqrt{\eta^2 - \nu^2\xi^2}}\right) s(t,\sigma R_2(\xi)), \quad t>0,\ \xi \in \mathbb{R}. \tag{5.21}$$

Since ℓ is 1-regular and θ-sectorial with $\theta < \frac{\pi}{2}$, it follows from representation (5.18) that

$$s(t,\sigma\nu) = -\int_0^\infty e^{-\nu\tau}\partial_\tau W_{\ell,\sigma}(t,\tau)d\tau, \quad t>0,\ Re(\nu)>0.$$

We note that for all $\xi \in \mathbb{R}$, the real part of both $R_1(\xi)$ and $R_2(\xi)$ is positive. Hence, it follows from (5.21) that

$$\widetilde{U}_k(t,\xi) = -\int_0^\infty \frac{e^{-\tau\eta}}{2}\left[\left(1 + \frac{\eta}{\sqrt{\eta^2-\nu^2\xi^2}}\right)e^{\tau\sqrt{\eta^2-\nu^2\xi^2}} + \left(1 - \frac{\eta}{\sqrt{\eta^2-\nu^2\xi^2}}\right)e^{-\tau\sqrt{\eta^2-c^2\xi^2}}\right]\partial_\tau W_{\ell,\sigma}(t,\tau)d\tau,$$

for all $t>0$ and $\xi \in \mathbb{R}$.

On the other hand, it is well-known that $\widetilde{P}(\tau,\xi)$ the Fourier transform of $P(\tau,x)$, see [86], is given by

$$\widetilde{P}(\tau,\xi) = \frac{e^{-\tau\eta}}{2}\left[\left(1 + \frac{\eta}{\sqrt{\eta^2-\nu^2\xi^2}}\right)e^{\tau\sqrt{\eta^2-\nu^2\xi^2}} + \left(1 - \frac{\eta}{\sqrt{\eta^2-\nu^2\xi^2}}\right)e^{-\tau\sqrt{\eta^2-\nu^2\xi^2}}\right],$$

which implies that

$$\widetilde{U}_k(t,\xi) = -\int_0^\infty \widetilde{P}(\tau,\xi)\partial_\tau W_{\ell,\sigma}(t,\tau)d\tau, \quad t>0,\ \xi\in\mathbb{R}.$$

Now, by taking the inverse Fourier transform, we have that $U_k(t,x)$ is given by (5.20). $\square$

***Remark* 5.5.** *The constant $\sigma > 0$ arising in the representation (5.20) can be chosen arbitrarily. Indeed, all the analysis can be made by taking $\sigma = 1$. However, as we will show by means of several examples, this constant plays a crucial role for relating $U_k(t,x)$ with the distribution of some concrete stochastic processes that appear in the context of subdiffusion theory.*

Proposition 5.4. *Let $(k,\ell) \in (\mathcal{PC})$. The fundamental solution of (1.10) can be interpreted as the distribution of a process of the form $T(|\mathcal{W}(t)|)$, $t>0$, where $T(t)$ is the telegraph process and $|\mathcal{W}(t)|$ is a random time whose law is $2V_{2k}(t,\tau)$ with $t>0$ and $\tau>0$, where $V_{2k}(\cdot,\cdot)$ is the solution to the nonlocal equation*

$$\partial_t^2(k*k*(u(\cdot,x)-\delta_0(x))(t) = \sigma^2\partial_x^2 u(t,x), \quad t>0,\ x\in\mathbb{R},$$

subject to the initial conditions

$$u(0,x) = \delta_0(x), \quad \text{and} \quad \partial_t u(t,x)\big|_{t=0} = 0,$$

where the initial condition imposed in the first derivative initial be considered only when exists.

Proof. The proof is a direct consequence of Theorem 5.4, Remark 5.5 and Proposition 5.3. $\square$

Now we focus our attention in analyzing two concrete pairs of functions $(k,\ell) \in (\mathcal{PC})$ that satisfy the conditions of Theorem 5.4 and Proposition 5.3. This will allows us to identify the distribution of the corresponding process $|\mathcal{W}(t)|$ with the distribution of some processes that are of interest in subdiffusion theory.

Time fractional case

Let $\alpha \in (0,1)$ and consider the pair $(k,\ell) = (g_{1-\alpha}, g_\alpha) \in (\mathcal{PC})$. In this case, it is well-known that

$$\widehat{\ell}(\lambda) = \frac{1}{\lambda^\alpha}, \quad \lambda > 0.$$

Therefore, a direct calculation shows that

$$\left|\lambda \frac{d}{d\lambda}\widehat{\ell}(\lambda)\right| \le \alpha|\widehat{\ell}(\lambda)|, \text{ for } Re(\lambda) > 0.$$

By Remark 2.9, this implies that ℓ is $\frac{\alpha\pi}{2}$-sectorial. Since $\alpha \in (0,1)$, this implies that ℓ is θ-sectorial with $\theta < \frac{\pi}{2}$. Thus, according to Theorem 5.4, for all $\alpha \in (0,1)$, the distribution of the process $X_\alpha(t)$ coincides with the distribution of a telegraph process $T(t)$ with a random time $|\mathcal{W}(t)|$, $t > 0$, where $|\mathcal{W}(t)|$ is a stochastic process whose law is $2V_{2\alpha}(t,x)$ with $t > 0$ and $x > 0$, where $V_{2\alpha}(t,x)$ is the solution to the time-fractional wave equation

$$\partial_t^2(g_{2-2\alpha} * (u(\cdot,x) - \delta_0(x))(t) = \sigma^2\partial_x^2 u(t,x), \quad t > 0,\, x \in \mathbb{R}, \tag{5.22}$$

together with the initial conditions

$$u(0,x) = \delta_0(x), \quad \text{and} \quad \partial_t u(t,x)|_{t=0} = 0, \quad x \in \mathbb{R}, \tag{4.9a}$$

whenever $\alpha \in (1/2, 1]$, and

$$u(0,x) = \delta_0(x), \quad x \in \mathbb{R}, \tag{4.9b}$$

if $\alpha \in (0, 1/2]$.

In what follows, we show that for some values of $\alpha \in (0,1)$ and an appropriate choice of $\sigma > 0$, the process $\mathcal{W}(t)$ can be identified in a more explicit manner. Our first example was first presented by Orsingher and Beghin [86, Section 4].

Example 5.7. *If $\alpha = \frac{1}{2}$ and $\sigma^2 = \frac{1}{2}$, then the distribution of $X_\alpha(t)$ coincides with the distribution of the process $T(|B(t)|)$, $t > 0$, where $T(t)$ is the classical telegraph process and $|B(t)|$ is the reflected Brownian motion.*

Proof. By definition of the Riemann-Liouville fractional derivative, if $\alpha = \frac{1}{2}$ and $\sigma^2 = \frac{1}{2}$, the equation (5.22) takes the form of the classical diffusion equation

$$\partial_t v(t,x) = \frac{1}{2}\,\partial_x^2 v(t,x), \quad t > 0,\; x \in \mathbb{R}, \tag{5.23}$$

whose fundamental solution is given by the heat kernel

$$H(t,x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{|x|^2}{2t}\right), \quad t > 0,\; x \in \mathbb{R}. \tag{5.24}$$

It is well-known that $2H(t,x)$ for $t > 0$ and $x > 0$, coincides with the probability density function of the reflected Brownian motion $|B(t)|$, $t > 0$. Consequently, the distribution of the process $X_{\frac{1}{2}}(t)$ coincides with that of the composition of the telegraph process and the reflected Brownian motion $|B(t)|$, $t \ge 0$. $\square$

Example 5.8. *If $\alpha = \frac{1}{4}$ and $\sigma^2 = 2^{-\frac{3}{2}}$, then the distribution of $X_\alpha(t)$ coincides with the distribution of the process $T(|B_1(|B_2(t)|)|)$, $t > 0$, where $T(t)$ is the classical telegraph process and B_1, B_2 are two independent Brownian motions.*

Proof. If $\alpha = \frac{1}{4}$ and $\sigma^2 = 2^{-\frac{3}{2}}$, then the equation (5.22) takes the form

$$\partial_t(k_1 * (v(\cdot,x) - \delta_0(x))(t) = \sigma^2\,\partial_x^2 v(t,x), \quad t > 0,\; x \in \mathbb{R}, \tag{5.25}$$

where $k_1 = g_{1/2}$. Since k_1 is a function of type $(\mathcal{PC})$, this equation fits in the theory of subdiffusion equations. To apply the theory developed in [56], we denote by $V_{\frac{1}{2}}(t,x)$ the fundamental solution of

(5.26). Following a similar procedure to that introduced in [56, Section 2], it can be proved that $V_{\frac{1}{2}}(t,x)$ can be represented as follows

$$V_{\frac{1}{2}}(t,x) = -\int_0^\infty H(\tau,x)\partial_\tau W_{\ell_1,\sigma_1}(t,\tau)d\tau, \quad t>0,\, x\in\mathbb{R},$$

where $H(\cdot,\cdot)$ is the heat kernel given in (5.24), and $W_{\ell_1,\sigma_1}(\cdot,\cdot)$ stands for the propagation function associated to ℓ_1 and σ_1 with $\ell_1 = g_{1/2}$ and $\sigma_1 := 2\sigma^2 = 2^{-\frac{1}{2}}$. This implies that the distribution of $X_{\frac{1}{4}}(t)$ coincides with the distribution of a process of the form

$$T(|B_1(|\mathcal{W}_1(t)|)|), \quad t\geq 0,$$

where $T(t)$ is the telegraph process, B_1 is a Brownian motion, and $|\mathcal{W}_1(t)|$ is a random time whose law is given by $\partial_\tau W_{\ell_1,\sigma_1}(t,\tau)$ for $t\geq 0$ and $\tau>0$. It follows from Proposition (5.3) that

$$\partial_\tau W_{\ell_1,\sigma_1}(t,\tau) = 2V_{2k_1}(t,\tau), \quad t>0, \tau>0,$$

where $V_{2k_1}(\cdot,\cdot)$ denotes the solution of the following evolution equation

$$\partial_t^2(k_1 * k_1 * (v(\cdot,x)-\delta_0(x))(t) = \sigma_1^2\partial_x^2 v(t,x), \quad t>0,\ x\in\mathbb{R}.$$

A direct computation shows that the preceding is equivalent to the classical diffusion equation

$$\partial_t v(t,x) = \frac{1}{2}\partial_x^2 v(t,x), \quad t>0,\ x\in\mathbb{R}.$$

From this point, the proof follows the same lines of those in the Example 5.7. Thus, we can prove that the distribution of $X_{\frac{1}{4}}(t)$ coincides with the distribution of a process of the form

$$T(|B_1(|B_2(t)|)|), \quad t\geq 0,$$

where $T(t)$ is the classical telegraph process and B_1, B_2 are two independent Brownian motions. □

Example 5.9. *If $\alpha = \frac{1}{8}$ and $\sigma^2 = 2^{-\frac{7}{4}}$, then the distribution of $X_\alpha(t)$ coincides with the distribution of the process $T(|B_1(|B_2(|B_3(t)|)|)|)$, $t>0$, where $T(t)$ is the classical telegraph process and B_1, B_2, B_3 are two independent Brownian motions.*

Proof. If $\alpha = \frac{1}{8}$ and $\sigma^2 = 2^{-\frac{7}{4}}$, then the equation (5.22) takes the form

$$\partial_t(k_2 * (v(\cdot,x)-\delta_0(x))(t) = \sigma^2\,\partial_x^2 v(t,x), \quad t>0,\ x\in\mathbb{R}, \tag{5.26}$$

where $k_2 = g_{1-1/4}$. Since k_2 is a function of type $(\mathcal{PC})$, this equation fits in the theory of subdiffusion equations. To apply the theory developed in [56], we denote by $V_{\frac{1}{4}}(t,x)$ the fundamental solution of (5.26). By applying the procedure introduced in [56, Section 2], we know that the Fourier transform of $V_{\frac{1}{4}}$ can be written as

$$\widetilde{V}_{\frac{1}{4}}(t,\xi) = -\int_0^\infty e^{-\frac{\tau\xi}{2}}\partial_\tau W_{\ell_2,\sigma_2}(t,\tau)d\tau, \quad t>0,\, x\in\mathbb{R},$$

where $W_{\ell_2,\sigma_2}(\cdot,\cdot)$ stands for the propagation function associate to ℓ_2 and σ_2 with $\ell_2 = g_{\frac{1}{4}}$ and $\sigma_2 := 2\sigma^2 = 2^{-\frac{3}{4}}$. This implies that the distribution of $X_{\frac{1}{4}}(t)$ coincides with the distribution of a process of the form

$$T(|B_1(|\mathcal{W}_2(t)|)|), \quad t\geq 0,$$

where $T(t)$ is the telegraph process, B_1 is a Brownian motion, and $|\mathcal{W}_2(t)|$ is a random time whose law is given by $\partial_\tau W_{\ell_2,\sigma_2}(t,\tau)$ for $t\geq 0$ and $\tau>0$. It follows from Proposition 5.3 that

$$\partial_\tau W_{\ell_2,\sigma_2}(t,\tau) = 2V_{2k_2}(t,\tau), \quad t>0, \tau>0,$$

where $V_{2k_2}(\cdot,\cdot)$ denotes the solution of the following evolution equation

$$\partial_t^2(k_2 * k_2 * (v(\cdot,x)-\delta_0(x))(t) = \sigma_2^2\partial_x^2 v(t,x), \quad t>0,\ x\in\mathbb{R}.$$

A direct computation shows that the preceding equation is equivalent to

$$\partial_t(k_1 * (v(\cdot,x)-\delta_0(x))(t) = \sigma_2^2\partial_x^2 v(t,x), \quad t>0,\ x\in\mathbb{R}, \tag{5.27}$$

where $k_1 = g_{1-\frac{1}{2}}$. According to [56, Section 2], the Fourier transform of the solution of (5.27), $V_{\frac{1}{2}}(t,x)$, can be written as

$$\widetilde{V}_{\frac{1}{2}}(t,\xi) = -\int_0^\infty e^{-\frac{\tau\xi}{2}}\partial_\tau W_{\ell_1,\sigma_1}(t,\tau)d\tau, \quad t>0,\, x\in\mathbb{R}, \tag{5.28}$$

where $W_{\ell_1,\sigma_1}(\cdot,\cdot)$ stands for the propagation function associate to ℓ_1 and the constant σ_1, with $\ell_1 = g_{\frac{1}{2}}$ and $\sigma_1 := 2(2\sigma^2)^2 = 2^{-\frac{1}{2}}$. This implies that the distribution of $X_{\frac{1}{4}}(t)$ coincides with the distribution of a process of the form

$$T(|B_1(|B_2(|\mathcal{W}_1(t)|)|)|), \quad t\geq 0,$$

where $T(t)$ is the telegraph process, B_1, B_2 are Brownian motions, and $|\mathcal{W}_1(t)|$ is a random time whose law is given by $\partial_\tau W_{\ell_1,\sigma_1}(t,\tau)$ for $t\geq 0$ and $\tau>0$. It follows from Proposition (5.3) that

$$\partial_\tau W_{\ell_1,\sigma_1}(t,\tau) = 2V_{2k_1}(t,\tau), \quad t>0, \tau>0,$$

where $V_{2k_1}(\cdot,\cdot)$ denotes the solution of the following evolution equation

$$\partial_t^2(k_1 * k_1 * (v(\cdot,x)-\delta_0(x))(t) = \sigma_1^2\partial_x^2 v(t,x), \quad t>0,\ x\in\mathbb{R}.$$

Finally, a direct computation shows that the preceding is equivalent to the classical diffusion equation

$$\partial_t v(t,x) = \frac{1}{2}\partial_x^2 v(t,x), \quad t>0,\ x\in\mathbb{R}.$$

From this point, the proof follows the same lines of those in the Example 5.7. Thus, we can prove that the distribution of $X_{\frac{1}{4}}(t)$ coincides with the distribution of a process of the form

$$T(|B_1(|B_2(|B_3(t)|)|)|), \quad t\geq 0,$$

where $T(t)$ is the classical telegraph process and B_1, B_2, B_3 are three independent Brownian motions. □

Beghin and Orsingher [87, Remark 3.5] extended the preceding result for $\alpha = \frac{1}{2^n}$ and $\sigma^2 = 2^{\frac{1}{2^{n-1}}-2}$ with $n \in \{1,2,3\cdots\}$. This result is established in the following example.

Example 5.10. *Let $n\in\mathbb{N}$. If $\alpha = \frac{1}{2^n}$ and $\sigma^2 = 2^{\frac{1}{2^{n-1}}-2}$ then the distribution of $X_\alpha(t)$ coincides with the distribution of the process*

$$T(|B_1(|B_2(|\cdots|B_{n-1}(|B_n(t)|)|)\cdots|)|), \quad t>0,$$

where $T(t)$ is the classical telegraph process and $B_1, B_2\cdots B_n$ are (n) independent Brownian motions.

Proof. Let $n\in\mathbb{N}$. Let us denote by k_n the function $k_n = g_{1-1/2^n}$. We recall that if $\alpha = \frac{1}{2^n}$ and $\sigma^2 = 2^{\frac{1}{2^{n-1}}-2}$, then the distribution of $X_\alpha(t)$ coincides with the law of a process of the form

$$T(|\mathcal{W}_n(t)|), \quad t\geq 0,$$

where $|\mathcal{W}_n(t)|$ is a random time whose law is given by $2V_{2k_n}(t,\tau)$ with $t\geq 0$ and $\tau>0$, where $V_{2k_n}(\cdot,\cdot)$ is the fundamental solution of the equation

$$\partial_t^2(k_n * k_n * (v(\cdot,x)-\delta_0(x))(t) = \sigma^2\,\partial_x^2 v(t,x), \quad t>0,\ x\in\mathbb{R}, \tag{5.29}$$

where δ_0 plays the role of the initial condition. By a straightforward calculation, we note that $(k_n * k_n) = (1 * k_{n-1})$, which implies that Equation (5.29) can be rewritten as follows

$$\partial_t (k_{n-1} * (v(\cdot, x) - \delta_0(x))(t) = \sigma^2 \, \partial_x^2 v(t,x), \quad t > 0, \; x \in \mathbb{R}. \tag{5.30}$$

It follows from [87, Theorem 2.2] that the fundamental solution of (5.30) coincides with the law of the n-times iterated Brownian motion

$$\mathcal{I}_n(t) = B_1(|B_2(...|B_n(t)|...)|), \quad t > 0,$$

where B_j with $j \in \{1, \cdots, n\}$ are independent Brownian motions. Therefore, the distribution of $\mathcal{W}_n(t)$ coincides with the distribution of $\mathcal{I}_n(t)$. This in turn implies that

$$X_\alpha(t) \overset{d}{\equiv} T(|B_1(|B_2(| \cdots |B_{n-1}(|B_n(t)|)|) \cdots |)|), \quad t > 0,$$

where $T(t)$ is the telegraph process and $B_1, B_2, \cdots, B_n$ are n independent standard Brownian motions. □

As we have mentioned before, both Example 5.7 and Example 5.10 were already present in the literature. However, our method is versatile enough for allowing us to obtain new examples.

Example 5.11. *If $\alpha = \frac{1}{3}$ and $\sigma^2 = 1$, then distribution of $X_\alpha(t)$ coincides with the distribution of the process $T(|A(t)|)$, $t > 0$, where $T(t)$ is the classical telegraph process and $A(t)$ is a stochastic process whose law is given by $\frac{3}{2}V(t,\tau)$, for $t > 0$ and $\tau > 0$, where $V(\cdot,\cdot)$ is the fundamental solution of the following evolution equation*

$$\partial_t v(t,x) = \partial_x^3 v(t,x), \quad t > 0, \; x \in \mathbb{R}. \tag{5.31}$$

Proof. According to Proposition 5.4, we know that if $\alpha = \frac{1}{3}$ and $\sigma^2 = 1$, then the distribution of $X_\alpha(t)$ coincides with the law of a process of the form

$$T(|\mathcal{W}(t)|), \quad t \geq 0,$$

where $|\mathcal{W}(t)|$ is a random time whose law is given by $2v(t,\tau)$ with $t \geq 0$ and $\tau > 0$, where $v(\cdot,\cdot)$ is the fundamental solution of the equation

$$\partial_t^2 (g_{2-\frac{2}{3}} * (v(\cdot, x) - \delta_0(x))(t) = \partial_x^2 v(t,x), \quad t > 0, \; x \in \mathbb{R}, \tag{5.32}$$

where δ_0 plays the role of the initial condition. By definition of time-fractional derivative in the sense of Riemann-Liouville, we have that the equation (5.32) is equivalent to

$$\partial_t^{2/3} (v(\cdot, x) - \delta_0(x))(t) = \partial_x^2 v(t,x), \quad t > 0, \; x \in \mathbb{R}. \tag{5.33}$$

Let us denote by $v_{2/3}(\cdot,\cdot)$ the solution of (5.33). It has been established in [87, Remark 4.1] that

$$v_{2/3}(t,x) = \frac{3}{2} V(t,|x|), \quad t > 0, \; x \in \mathbb{R},$$

where $V(\cdot,\cdot)$ is the fundamental solution of (5.31). Since $\frac{3}{2}V(t,\tau)$ with $t > 0$ and $\tau > 0$ is the probability density function of the process $A(t)$, according to Remark 5.5, we conclude that $2v_{2/3}(t,\tau)$ with $t > 0$ and $\tau > 0$ is the distribution of the process $|A(t)|$, $t > 0$, which completes the proof. □

***Remark* 5.6.** *Some of the features of $A(t)$ have been analyzed by several researchers in the last decades. For instance, conditional distributions of sojourn times for this type of processes has been analyzed by Nikitin and Orsingher [84], unconditional sojourn laws have been studied by Orsingher [85], the asymptotic behavior of the fundamental solution is analyzed by Accetta and Orsingher [1] and some specific connections with complex-valued walks are examined by Hochberg and Orsingher [47].*

Example 5.12. *If $\alpha = \frac{1}{6}$ and $\sigma^2 = \frac{1}{2}$, then the distribution of $X_\alpha(t)$ coincides with the distribution of the process $T(|B(|A(t)|)|)$, $t > 0$, where $T(t)$ is the classical telegraph process, $B(t)$ is the Brownian motion, and $A(t)$ is the process described in Example (5.11).*

Proof. By Proposition 5.4, we know that if $\alpha = \frac{1}{6}$ and $\sigma^2 = \frac{1}{2}$, then the distribution of $X_\alpha(t)$ coincides with the law of a process of the form

$$T(|\mathcal{W}(t)|), \quad t \geq 0,$$

where $|\mathcal{W}(t)|$ is a random time whose law is given by $2v(t,\tau)$ with $t \geq 0$ and $\tau > 0$, where $v(\cdot,\cdot)$ is the fundamental solution of the equation

$$\partial_t^2(g_{2-\frac{1}{3}} * (v(\cdot,x) - \delta_0(x))(t) = \frac{1}{2}\partial_x^2 v(t,x), \quad t > 0, \ x \in \mathbb{R}, \tag{5.34}$$

where δ_0 plays the role of the initial condition. By definition of time-fractional derivative in the sense of Riemann-Liouville, we have that the equation (5.34) is equivalent to

$$\partial_t^{1/3}(v(\cdot,x) - \delta_0(x))(t) = \frac{1}{2}\,\partial_x^2 v(t,x), \quad t > 0, \ x \in \mathbb{R}. \tag{5.35}$$

Let us denote by $v_{1/3}(\cdot,\cdot)$ the solution of (5.35). Following the same ideas that those introduced in [56, Section 2] it can be proved that

$$v_{1/3}(t,x) = -\int_0^\infty H(\tau,x)\partial_\tau W_{\ell_1,\sigma_1}(t,\tau)d\tau, \quad t > 0, \ x \in \mathbb{R},$$

where $H(t,x)$ is the heat kernel defined in (5.24) and $W_{\ell_1,\sigma_1}(\cdot,\cdot)$ is the propagation function associated to ℓ_1 and σ_1, with $\ell_1 = g_{2/3}$ and $\sigma_1^2 = 2\sigma^2 = 1$. This shows that in this case the distribution of $X_\alpha(t)$ coincides with the distribution of a process of the form

$$T(|B(|\mathcal{W}_1(t)|)|), \quad t > 0,$$

where $B(t)$ is the Brownian motion and $|\mathcal{W}_1(t)|$ is a random time whose law is given by $\partial_\tau(-W_{\ell_1,\sigma_1}(t,\tau))$ for $t > 0$ and $\tau > 0$. According to Proposition 5.3, we have that

$$\partial_\tau(-W_{\ell_1,\sigma_1}(t,\tau)) = 2V_{2k_1}(t,\tau), \quad t > 0, \ \tau > 0,$$

where $k_1 = g_{2/3}$ and $V_{2k_1}(\cdot,\cdot)$ is the fundamental solution of the following evolution equation

$$\partial_t^2(k_1 * k_1 * (v(\cdot,x) - \delta_0(x))(t) - v(t,x) = 0, \quad t > 0, \ x \in \mathbb{R}, \tag{5.36}$$

where δ_0 stands for the initial value. By definition of the time-fractional derivative, we observe that the equation (5.36) is equivalent to (5.33). Therefore, the distribution of $X_\alpha(t)$ coincides with the distribution of a process of the form

$$T(|B(|A(t)|)|), \quad t > 0,$$

where $A(t)$ is the stochastic process described in Example 5.11. □

Example 5.13. *Let $n \in \mathbb{N}$. If $\alpha = \frac{1}{3\cdot 2^n}$ and $\sigma^2 = 2^{\frac{1}{2^n}-2}$ then distribution of $X_\alpha(t)$ coincides with the distribution of the process*

$$T(|B_1(|B_2(|\cdots|B_n(|A(t)|)|)\cdots|)|), \quad t > 0,$$

where $T(t)$ is the classical telegraph process, $B_1, B_2 \cdots B_n$ are n independent Brownian motions and $A(t)$ is the stochastic process described in Example 5.11.

Proof. To prove this result we need to introduce the following notation. For $n \in \mathbb{N}_0$, we denote by k_n the function $k_n = g_{1-1/3\cdot 2^n}$, and by ℓ_n the function $\ell_n = g_{1/3\cdot 2^n}$.

It follows from Proposition 5.4 that if $\alpha = \frac{1}{3\cdot 2^n}$ and $\sigma^2 = 2^{\frac{1}{2^{n-1}}-2}$, then the distribution of $X_\alpha(t)$ coincides with the law of a process of the form

$$T(|\mathcal{W}_n(t)|), \quad t \geq 0,$$

where $T(t)$ is the telegraph process and $|\mathcal{W}_n(t)|$ is a random time whose law is given by $2V_{2k_n}$, where V_{2k_n} is the fundamental solution of the following evolution equation

$$\partial_t^2(k_n * k_n * (v(\cdot,x) - \delta_0(x)))(t) = \sigma_n^2\,\partial_x^2 v(t,x), \quad t > 0, \ x \in \mathbb{R}, \tag{5.37}$$

where $\sigma_n = \sigma$. We observe that the equation (5.37) takes the form

$$\partial_t(k_{n-1} * (v(\cdot,x) - \delta_0(x))(t) = \sigma_n^2\, \partial_x^2 v(t,x), \quad t>0,\ x\in\mathbb{R}. \tag{5.38}$$

This equation fits in the theory of subdiffusion equations. To apply the theory developed in [56], we denote by $V_{n-1}(\cdot,\cdot)$ the fundamental solution of (5.38). Following a similar procedure to that introduced in [56, Section 2], it can be proved that $V_{n-1}(t,x)$ can be represented as follows

$$V_{n-1}(t,x) = -\int_0^\infty H(\tau,x)\partial_\tau W_{\ell_{n-1},\sigma_{n-1}}(t,\tau)d\tau, \quad t>0,\, x\in\mathbb{R},$$

where $H(\cdot,\cdot)$ is the heat kernel given in (5.24), and $W_{\ell_{n-1},\sigma_{n-1}}(\cdot,\cdot)$ stands for the propagation function associated to ℓ_{n-1} and σ_{n-1}, with $\sigma_{n-1} := 2\sigma_n^2 = 2^{\frac{1}{2^{n-1}}-1}$. This implies that the distribution of $X_{\frac{1}{3\cdot 2^n}}(t)$ coincides with the distribution of a process of the form

$$T(|B_1(|\mathcal{W}_{n-1}(t)|)|), \quad t\geq 0,$$

where $\left|\mathcal{W}_{n-1}(t)\right|$ is a random time whose law is given by $\partial_\tau(-W_{\ell_{n-1},\sigma_{n-1}}(t,\tau))$ for $t\geq 0$ and $\tau>0$. It follows from Proposition 5.3 that

$$\partial_\tau(-W_{\ell_{n-1},\sigma_{n-1}}(t,\tau)) = 2V_{2k_{n-1}}(t,\tau), \quad t>0, \tau>0,$$

where $V_{2k_{n-1}}(\cdot,\cdot)$ stands for the fundamental solution of the following evolution equation

$$\partial_t^2(k_{n-1} * k_{n-1} * (v(\cdot,x) - \delta_0(x))(t) = \sigma_{n-1}^2 \partial_x^2 v(t,x), \quad t>0,\ x\in\mathbb{R}.$$

A direct computation shows that the preceding equation is equivalent to the following evolution equation

$$\partial_t(k_{n-2} * (v(\cdot,x) - \delta_0(x))(t) = \sigma_{n-1}^2\, \partial_x^2 v(t,x), \quad t>0,\ x\in\mathbb{R}. \tag{5.39}$$

If $n=2$, then $\sigma_{n-1}^2 = \frac{1}{2}$ and the equation (5.39) corresponds to the subdiffusion equation

$$\partial_t^{\frac{1}{3}}(v(\cdot,x) - \delta_0(x))(t) = \frac{1}{2}\partial_x^2 v(t,x), \quad t>0,\ x\in\mathbb{R}.$$

In such a case, the folded fundamental solution of the preceding equation coincides with the distribution of a process $|B_2(|A(t)|)|$, for $t>0$, where $B_2(t)$ is a standard Brownian motion and $A(t)$ has been described in Example 5.11. Therefore, the distribution of $X_\alpha(t)$ coincides with the distribution of the process

$$T(|B_1(|B_2(|A(t)|)|)|), \quad t\geq 0.$$

If $n\geq 3$, the fundamental solution will be denoted by $V_{n-2}(\cdot,\cdot)$, which can be represented as follows

$$V_{n-2}(t,x) = -\int_0^\infty H(\tau,x)\partial_\tau W_{\ell_{n-2},\sigma_{n-2}}(t,\tau)d\tau, \quad t>0,\, x\in\mathbb{R},$$

where $H(\cdot,\cdot)$ is the heat kernel given in (5.24), and $W_{\ell_{n-2},\sigma_{n-2}}(\cdot,\cdot)$ stands for the propagation function associated to ℓ_{n-2} and σ_{n-2}, with $\sigma_{n-2} := 2\sigma_{n-1}^2 = 2^{\frac{1}{2^{n-2}}-1}$. This implies that the distribution of $X_{\frac{1}{3\cdot 2^n}}(t)$ coincides with the distribution of a process of the form

$$T(|B_1(|B_2(|\mathcal{W}_{n-2}(t)|)|)|), \quad t\geq 0,$$

where $\left|\mathcal{W}_{n-2}(t)\right|$ is a random time whose law is given by $\partial_\tau(-W_{\ell_{n-2},\sigma_{n-2}}(t,\tau))$ for $t\geq 0$ and $\tau>0$. It follows from Proposition 5.3 that

$$\partial_\tau(-W_{\ell_{n-2},\sigma_{n-2}}(t,\tau)) = 2V_{2k_{n-2}}(t,\tau), \quad t>0, \tau>0,$$

where $V_{2k_{n-2}}(\cdot,\cdot)$ stands for the solution of the following evolution equation

$$\partial_t^2(k_{n-2} * k_{n-2} * (v(\cdot,x) - \delta_0(x))(t) = \sigma_n^2 \partial_x^2 v(t,x), \quad t>0,\ x\in\mathbb{R}.$$

A direct computation shows that the preceding is equivalent to the following evolution equation

$$\partial_t(k_{n-3} * (v(\cdot,x) - \delta_0(x))(t) = \sigma_{n-1}^2\, \partial_x^2 v(t,x), \quad t>0,\ x\in\mathbb{R}. \tag{5.40}$$

If $n=3$, then $\sigma_{n-2}^2 = \frac{1}{2}$, and the equation (5.40) corresponds to the subdiffusion equation

$$\partial_t^{\frac{1}{3}} v(t,x) = \frac{1}{2}\partial_x^2 v(t,x), \quad t>0,\ x\in\mathbb{R}.$$

Therefore, according to Example 5.12 the distribution of $X_\alpha(t)$ coincides with the distribution of the process of the form

$$T(|B_1(|B_2(|B_3(|A(t)|)|)|)|), \quad t\geq 0.$$

In general, for $n\in\mathbb{N}$ the procedure described above can be repeated n-times until we obtain the subdiffusion equation

$$\partial_t^{\frac{2}{3}} v(t,x) = \partial_x^2 v(t,x), \quad t>0,\ x\in\mathbb{R}.$$

Therefore, we have proved that the distribution of $X_{\frac{1}{3\cdot 2^n}}(t)$ agrees with the distribution of a process of the form

$$T(|B_1(|B_2(|\cdots|B_{n-1}(|B_n(|A(t)|)|)|)\cdots|)|), \quad t>0.$$

where $T(t)$ is the telegraph process and $B_1,\cdots B_{n-1}, B_n$ are n independent Brownian motions and $A(t)$ is the stochastic process described in Example 5.11.

□

Distributed-order case

Before presenting our following examples, we need to recall some facts about the so-called *ultra-slow diffusion equation*

$$\partial_t(k_d * (v(\cdot,x) - \delta_0(x)))(t) = \partial_x^2 v(t,x), \quad t>0,\ x\in\mathbb{R}, \tag{5.41}$$

where $(k_d,\ell_d)\in(\mathcal{PC})$ is the pair defined in Example 5.2. It has been established by Kochubei [60, Section 3] that the fundamental solution of (5.41) can be interpreted as the probability density function of a stochastic process $\mathcal{P}_d(t)$ for $t>0$. Further, it can be proved that the variance of $\mathcal{P}_d(t)$ grows (at infinity) like a logarithmic function, see [56, Section 2]. Later, Meerschaert, Nane and Vellaisamy [78, Section 5] have studied some other features of this process. In such a work, the authors have referred this process as *Poisson distributed-order process*. In this work, we will adopt the same nomenclature to refer the process $\mathcal{P}_d(t)$.

Let $\delta\in(0,1)$ and consider the pair $(k_\delta,\ell_\delta)\in(\mathcal{PC})$ whose Laplace transforms are given by

$$\widehat{k}_\delta(\lambda) = \begin{cases} \frac{1}{\lambda}\left(\frac{\lambda-1}{\log(\lambda)}\right)^\delta, & \lambda\neq 1\\ 1, & \lambda=1,\end{cases} \tag{5.42}$$

$$\widehat{\ell}_\delta(\lambda) = \begin{cases} \left(\frac{\log(\lambda)}{\lambda-1}\right)^\delta, & \lambda\neq 1\\ 1, & \lambda=1.\end{cases} \tag{5.43}$$

These pairs were recently introduced by Alegría and Pozo in [4, Corollary 3.10] or Chapter 4, and they are strongly related to the so-called ***ultra-slow diffusion processes***. We also point out that if $\delta=1$ then (k_δ,ℓ_δ) coincides with the pair (k_d,ℓ_d) defined in Example (5.2).

Proposition 5.5. *Let $\delta\in(0,1)$. Then ℓ_δ is a 1-regular and $\frac{\delta\pi}{2}$-sectorial function.*

$$\lambda\frac{d}{d\lambda}\widehat{\ell}_\delta(\lambda) = \frac{\delta}{(\lambda-1)^\delta}\left(1-\frac{\lambda}{\lambda-1}\log(\lambda)\right)\log(\lambda)^{\delta-1}, \quad \lambda>0,\ \lambda\neq 1,$$

Let us now define the function

$$h(\lambda) = \lambda\log(\lambda) - \lambda + 1, \quad \lambda>0.$$

We observe that h is decreasing in the interval $\lambda \in (0,1)$ and increasing in the interval $(1,\infty)$. Since h is nonnegative, $h(1) = 0$ and

$$\lim_{\lambda\to 0^+} g(\lambda) = 1,$$

we have that

$$-\left(1 - \frac{\lambda}{\lambda-1}\log(\lambda)\right) \leq \log(\lambda) \quad \text{if } \lambda > 1,$$

and

$$\left(1 - \frac{\lambda}{\lambda-1}\log(\lambda)\right) \leq -\log(\lambda) \quad \text{if } 0 < \lambda < 1.$$

Therefore, we conclude that

$$\left|\lambda \frac{d}{d\lambda}\widehat{\ell}_\delta(\lambda)\right| \leq \delta|\widehat{\ell}_\delta(\lambda)|, \quad \lambda > 0,\ \lambda \neq 1,$$

which implies that ℓ_δ is 1-regular. Since ℓ_δ is real-valued and 1-regular, it follows from Remark 2.9 that ℓ is θ-sectorial with $\theta = \frac{\delta\pi}{2} < \frac{\pi}{2}$.

Example 5.14. *If $\delta = \frac{1}{2}$ and $\sigma^2 = 1$, then the distribution of $X_{k_\delta}(t)$ coincides with the distribution of the process $T(|P_d(t)|)$, $t > 0$, where $T(t)$ is the classical telegraph process and $P_d(t)$ is the so-called distributed-order Poisson process.*

Proof. By proposition 5.4, we recall that if $\delta = \frac{1}{2}$ and $\sigma^2 = 1$, then the distribution of $X_{k_\delta}(t)$ coincides with the law of a process of the form

$$T(|\mathcal{W}(t)|), \quad t \geq 0,$$

where $|\mathcal{W}(t)|$ is a random time whose law is given by the folded solution of the following evolution equation

$$\partial_t^2(k_{\frac{1}{2}} * k_{\frac{1}{2}} * (v(\cdot,x) - \delta_0(x)))(t) - \partial_x^2 v(t,x) = 0, \quad t > 0,\ x \in \mathbb{R}. \tag{5.44}$$

By a Laplace transform argument, we note that

$$(k_{\frac{1}{2}} * k_{\frac{1}{2}})(t) = (1 * k_d)(t), \quad t > 0.$$

Therefore, Equation (5.44) is equivalent to (5.41). Thus, it follows from Remark 5.4 and Proposition 5.3 that the distribution of $X_{k_\delta}(t)$ coincides with the distribution of a process

$$T(|\mathcal{P}_d(t)|), \quad t > 0.$$

□

Example 5.15. *If $\delta = \frac{1}{2^n}$ and $\sigma^2 = 2^{\frac{1}{2^{n-1}}-1}$, then distribution of $X_{k_\delta}(t)$ coincides with the distribution of the process*

$$T(|B_1(|B_2 \cdots (|B_{n-1}(|P_d(t)|) \cdots |)|)|), \quad t > 0,$$

where $T(t)$ is the classical telegraph process, $B_1, B_2, \cdots, B_{n-1}$ are $(n-1)$ independent Brownian motions and $P_d(t)$ is the so-called distributed-order Poisson process, described in Example 5.14.

Proof. For $n = 1$, this result has been already proved in Example 5.14. In the proof of this result, we will denote by k_n and ℓ_n the functions whose Laplace transform are given by

$$\widehat{k}_n(\lambda) = \frac{1}{\lambda}\left(\frac{\lambda-1}{\log(\lambda)}\right)^{\frac{1}{2^n}}, \quad \text{and} \quad \widehat{\ell}_n(\lambda) = \left(\frac{\log(\lambda)}{\lambda-1}\right)^{\frac{1}{2^n}} \quad \lambda > 0.$$

In order to clarify our result, we consider first the case $n = 2$. By Proposition 5.4, the distribution of $X_{k_2}(t)$ coincides with the law of a process of the form

$$T(|\mathcal{W}_2(t)|), \quad t \geq 0,$$

where $|\mathcal{W}_2(t)|$ is a random time whose law is given by the folded solution of the following evolution equation

$$\partial_t^2(k_2 * k_2 * (v(\cdot,x) - \delta_0(x)))(t) - \sigma_2^2\,\partial_x^2 v(t,x) = 0, \quad t > 0,\ x \in \mathbb{R}, \tag{5.45}$$

where $\sigma_2 = \sigma$. By a Laplace transform argument, we note that

$$(k_2 * k_2)(t) = (1 * k_1)(t), \quad t > 0,$$

which implies that Equation (5.45) is equivalent to the following evolution equation

$$\partial_t (k_1 * (v(\cdot, x) - \delta_0(x)))(t) - \sigma_2^2 \, \partial_x^2 v(t, x) = 0, \quad t > 0, \; x \in \mathbb{R}. \tag{5.46}$$

It has been proved in [4, Corollary 3.13] that k_1 is a kernel of type $(\mathcal{PC})$. Therefore, by applying the theory developed in [56], we can prove that

$$V_1(t, x) = -\int_0^\infty H(\tau, x) \partial_\tau W_{\ell_1, \sigma_1}(t, \tau) d\tau, \quad t > 0, \; x \in \mathbb{R},$$

where $V_1(\cdot, \cdot)$ is the fundamental solution of (5.46), $\sigma_1 = 2\sigma_2^2$ and $W_{\ell_1, \sigma_1}(t, \tau)$ is the propagation function associated to ℓ_1 and σ_1. This shows that the distribution of $X_{k_2}(t)$ coincides with the distribution of a process of the form

$$T(|B_1(|\mathcal{W}_1(t)|)|), \quad t > 0,$$

where $|\mathcal{W}_1(t)|$ is a random time whose law is given by $\partial_\tau(-W_{\ell_1, \sigma_1}(t, \tau))$ for $t \geq 0$ and $\tau > 0$. It follows from Proposition 5.3 that

$$\partial_\tau(-W_{\ell_1, \sigma_1}(t, \tau)) = 2V_{2k_1}(t, \tau), \quad t > 0, \tau > 0,$$

where $V_{2k_1}(\cdot, \cdot)$ stands for the fundamental solution of the following evolution equation

$$\partial_t^2 (k_1 * k_1 * (v(\cdot, x) - \delta_0(x))(t) = \sigma_1^2 \partial_x^2 v(t, x), \quad t > 0, \; x \in \mathbb{R}. \tag{5.47}$$

Since $k_1 * k_1 = 1 * k_d$ and $\sigma_1^2 = 1$, we note that (5.47) is equivalent to the following evolution equation

$$\partial_t (k_d * (v(\cdot, x) - \delta_0(x))(t) = \partial_x^2 v(t, x), \quad t > 0, \; x \in \mathbb{R}. \tag{5.48}$$

Therefore, the distribution of $X_{k_2}(t)$ coincides with the distribution of a process of the form

$$T(|B_1(|\mathcal{P}_d(t)|)|), \quad t > 0.$$

In general, for $n \in \mathbb{N}$, the procedure described above can be repeated $(n-1)$-times until we obtain the subdiffusion equation

$$\partial_t (k_d * (v(\cdot, x) - \delta_0(x))(t) = \partial_x^2 v(t, x), \quad t > 0, \; x \in \mathbb{R}.$$

This implies that the distribution of $X_{k_n}(t)$ agrees with the distribution of a process of the form

$$T(|B_1(|B_2(|\cdots|B_{n-1}(|\mathcal{P}_d(t)|)|)\cdots|)|), \quad t > 0,$$

where $T(t)$ is the telegraph process and $B_1, \cdots B_{n-1}$ are $(n-1)$ independent Brownian motions and $\mathcal{P}_d(t)$ is the so-called distributed order Poisson process described in Example 5.14.

□

Kac [54] has established that if $c^2/\mu \to 1$ as $\mu, c \to \infty$, then the probability density function $P(t, x)$ converges to the heat kernel $H(t, x)$. As a direct consequence of Theorem 5.4, we prove a similar result in the nonlocal framework.

Corollary 5.1. *Let $(k, \ell) \in (\mathcal{PC})$ be such that ℓ is 1-regular and θ-sectorial with $\theta < \frac{\pi}{2}$. Then, the fundamental solution $U_k(\cdot, \cdot)$ of (1.10) converges to the fundamental solution $V_k(\cdot, \cdot)$ of the equation*

$$\partial_t (k * (v(\cdot, x) - \delta_0(x)) - \frac{1}{2} \partial_x^2 v(t, x) = 0, \quad t > 0, \; x \in \mathbb{R},$$

as $\mu, c \to \infty$ in such a way that $\frac{c^2}{\mu} \to 1$.

Proof. Since ℓ is 1-regular and θ-sectorial with $\theta < \frac{\pi}{2}$, it follows from Theorem 5.4 that $U_k(t,x)$ can be represented as follows

$$U_k(t,x) = -\int_0^\infty P(\tau,x)\partial_\tau W_{\ell,\sigma}(t,\tau)d\tau, \quad t>0,\ x\in\mathbb{R},$$

where $P(t,x)$ is the probability density function of the classical telegraph process, and $W_{\ell,\sigma}(\cdot,\cdot)$ is the propagation function associated to the function ℓ and σ. It follows from [54] that if $c^2/\mu \to 1$ as $\mu, c\to\infty$, then the probability density function $P(t,x)$ converges to the heat kernel $H(t,x)$. Therefore, by a direct application of the Dominated Convergence Theorem we obtain that

$$\lim_{\substack{\mu,c\to\infty\\ c^2/\mu\to 1}} U_k(t,x) = -\int_0^\infty H(\tau,x)\partial_\tau W_{\ell,\sigma}(t,\tau)d\tau, \quad t>0,\ x\in\mathbb{R}.$$

Following the same arguments developed in [56, Section 2], it can be established that the fundamental solution $V_k(\cdot,\cdot)$ can be represented as

$$V_k(t,x) = -\int_0^\infty H(\tau,x)\partial_\tau W_{\ell,\sigma}(t,\tau)d\tau, \quad t>0,\ x\in\mathbb{R},$$

which completes the proof. □

***Remark* 5.7.** *There are other general contexts where the pairs $(k,\ell)\in(\mathcal{PC})$ play a fundamental role. For example, this type of functions has been successfully exploited to study the so-called sub-diffusion processes, see [56, 93, 113] and references therein. Indeed, let us consider the so-called* sub-diffusion equation

$$\partial_t(k*(u(\cdot,x)-u_0(x)))(t) - \sigma^2\partial_x^2 u(t,x) = 0, \quad t>0,\, x\in\mathbb{R}, \tag{5.49}$$

subject to the initial conditions $u(0,x)=u_0(x)$, where k is a kernel of type $(\mathcal{PC})$ and u_0 is a given function. It has been proved, see [56], that the fundamental solution associated of the equation (5.49) can be viewed as a probability density function of a stochastic process $Y(t)$. The importance of this kind of processes is that they can be considered as a non-Markovian version of the Brownian motion, see e.g. [80, 87].

We point out that following the same scheme of the proof of Theorem 5.1, it can be proved that, for all $n\in\mathbb{N}$, the n-th moment of $Y(t)$ is given by the expression

$$M_n(t) = \begin{cases} 0, & \text{if } n=2m+1, \text{ for some } m\in\mathbb{N}_0,\\ (2m)!\,\sigma^{2m}\left(1*\ell^{*(m)}\right)(t), & \text{if } n=2m, \text{ for some } m\in\mathbb{N}, \end{cases}$$

where $\ell^{(*m)} = (\ell^{*(m-1)}*\ell)$ for $m>1$ and $\ell^{*(1)}=\ell$. Thus, following the same procedure developed in Theorem 5.1, it can be proved that these moments satisfy the Carleman condition. In addition, analogously to the proof of Theorem 5.2, we can find en explicit formula for the Moment generating function of the process $Y(t)$. Finally, making the corresponding adaptations to the proof of the Theorem 5.3, we can give the precise description of the asymptotic behavior of the moments of $Y(t)$. For the sake of the brevity of the text we omit the proofs.

Bibliography

[1] Accetta, G., and Orsingher, E. Asymptotic expansion of fundamental solutions of higher order heat equations. *Random Oper. Stochastic Equations 5*, 3 (1997), 217–226.

[2] Akhiezer, N. I. *The classical moment problem and some related questions in analysis.* Hafner Publishing Co., New York, 1965. Translated by N. Kemmer.

[3] Alegría, F., Poblete, V., and Pozo, J. C. Non-local in time telegraph equation and telegraph processes with random time. *Submitted for publication* (2022).

[4] Alegría, F., and Pozo, J. C. Non-local in time telegraph equations and very slowly growing variances. *Proc. Amer. Math. Soc. 149*, 5 (2021), 2067–2080.

[5] Arendt, W., Batty, C. J. K., Hieber, M., and Neubrander, F. *Vector-valued Laplace transforms and Cauchy problems*, second ed., vol. 96 of *Monographs in Mathematics.* Birkhäuser/Springer Basel AG, Basel, 2011.

[6] Bart, D., and Bart, J. *The Antique Wireless Assoc Review*, vol. 21. AWA, 2008.

[7] Bartlett, M. S. Some problems associated with random velocity. *Publ. Inst. Statis. Univ. Paris 6* (1957), 261–270.

[8] Bazhlekova, E. G. Subordination principle for fractional evolution equations. *Fract. Calc. Appl. Anal. 3*, 3 (2000), 213–230.

[9] Beghin, L., Macci, C., and Ricciuti, C. Random time-change with inverses of multivariate subordinators: governing equations and fractional dynamics. *Stochastic Process. Appl. 130*, 10 (2020), 6364–6387.

[10] Bergman, S., and Martin, W. T. A modified moment problem in two variables. *Duke Math. J. 6* (1940), 389–407.

[11] Bingham, N. H., Goldie, C. M., and Teugels, J. L. *Regular variation*, vol. 27 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 1989.

[12] Boas, Jr., R. P. On a generalization of the Stieltjes moment problem. *Trans. Amer. Math. Soc. 46* (1939), 142–150.

[13] Boas, Jr., R. P. The Stieltjes moment problem for functions of bounded variation. *Bull. Amer. Math. Soc. 45*, 6 (1939), 399–404.

[14] Bonaccorsi, S., D'Ovidio, M., and Mazzucchi, S. Probabilistic representation formula for the solution of fractional high-order heat-type equations. *J. Evol. Equ. 19*, 2 (2019), 523–558.

[15] Caffarelli, L., and Vazquez, J. L. Nonlinear porous medium flow with fractional potential pressure. *Arch. Ration. Mech. Anal. 202*, 2 (2011), 537–565.

[16] Cane, V. R. Diffusion models with relativity effects. In *Perspectives in probability and statistics (papers in honour of M. S. Bartlett on the occasion of his 65th birthday).* 1975, pp. 263–273.

[17] Cannarsa, P., and Sforza, D. Integro-differential equations of hyperbolic type with positive definite kernels. *J. Differential Equations 250*, 12 (2011), 4289–4335.

[18] Capiński, M., and Kopp, E. *Measure, integral and probability*, second ed. Springer Undergraduate Mathematics Series. Springer-Verlag London, Ltd., London, 2004.

[19] Carillo, S. Some remarks on the model of rigid heat conductor with memory: unbounded heat relaxation function. *Evol. Equ. Control Theory 8*, 1 (2019), 31–42.

[20] Carleman, T. Sur les fonctions quasi-analytiques, conférences faites au cinquième congres des mathématiciens scandinaves. *Helsinki 181* (1923), 196.

[21] Carleman, T. *Les functions quasi analytiques.* Gauthier-Villars, 1926.

[22] Carlone, R., Fiorenza, A., and Tentarelli, L. The action of Volterra integral operators with highly singular kernels on Hölder continuous, Lebesgue and Sobolev functions. *J. Funct. Anal. 273*, 3 (2017), 1258–1294.

[23] Cascaval, R. C., Eckstein, E. C., Frota, C. L., and Goldstein, J. A. Fractional telegraph equations. *J. Math. Anal. Appl. 276*, 1 (2002), 145–159.

[24] Cinlar, E. *Introduction to stochastic processes.* Courier Corporation, 2013.

[25] Clément, P., and Nohel, J. A. Abstract linear and nonlinear Volterra equations preserving positivity. *SIAM J. Math. Anal. 10*, 2 (1979), 365–388.

[26] Clément, P., and Nohel, J. A. Asymptotic behavior of solutions of nonlinear Volterra equations with completely positive kernels. *SIAM J. Math. Anal. 12*, 4 (1981), 514–535.

[27] Constantin, P. Euler equations, Navier-Stokes equations and turbulence. In *Mathematical foundation of turbulent viscous flows*, vol. 1871 of *Lecture Notes in Math.* Springer, Berlin, 2006, pp. 1–43.

[28] Constantin, P., Majda, A. J., and Tabak, E. Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar. *Nonlinearity 7*, 6 (1994), 1495–1533.

[29] Debrouwere, A. Solution to the Stieltjes moment problem in Gelfand-Shilov spaces. *Studia Math. 254*, 3 (2020), 295–323.

[30] Dell'Oro, F., Goubet, O., Mammeri, Y., and Pata, V. A semidiscrete scheme for evolution equations with memory. *Discrete Contin. Dyn. Syst. 39*, 10 (2019), 5637–5658.

[31] Dipierro, S., Palatucci, G., and Valdinoci, E. Dislocation dynamics in crystals: a macroscopic theory in a fractional Laplace setting. *Comm. Math. Phys. 333*, 2 (2015), 1061–1105.

[32] Estrada, R., and Vindas, J. General Stieltjes moment problems for rapidly decreasing smooth functions. *Studia Math. 238*, 3 (2017), 271–295.

[33] Feller, W. Completely monotone functions and sequences. *Duke Math. J. 5* (1939), 661–674.

[34] Feller, W. *An introduction to probability theory and its applications. Vol. II.* Second edition. John Wiley & Sons, Inc., New York-London-Sydney, 1971.

[35] Germann, N., Gurnon, A. K., Zhou, L., Cook, L., Beris, A., and Wagner, N. Validation of constitutive modeling of shear banding, threadlike wormlike micellar fluids. *Journal of Rheology 60* (09 2016), 983–999.

[36] Giacomin, G., and Lebowitz, J. L. Phase segregation dynamics in particle systems with long range interactions. I. Macroscopic limits. *J. Statist. Phys. 87*, 1-2 (1997), 37–61.

[37] Goldstein, S. On diffusion by discontinuous movements, and on the telegraph equation. *Quart. J. Mech. Appl. Math. 4* (1951), 129–156.

[38] Gorenflo, R., Kilbas, A. A., Mainardi, F., and Rogosin, S. V. *Mittag-Leffler functions, related topics and applications.* Springer Monographs in Mathematics. Springer, Heidelberg, 2014.

[39] Gripenberg, G., Londen, S.-O., and Staffans, O. *Volterra integral and functional equations*, vol. 34 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 1990.

[40] Gripenberg, G., and Norros, I. On the prediction of fractional Brownian motion. *J. Appl. Probab. 33*, 2 (1996), 400–410.

[41] Gurtin, M. E., and Pipkin, A. C. A general theory of heat conduction with finite wave speeds. *Arch. Rational Mech. Anal. 31*, 2 (1968), 113–126.

[42] HAMBURGER, H. Über eine Erweiterung des Stieltjesschen Momentenproblems. *Math. Ann. 82*, 1-2 (1920), 120–164.

[43] HAMBURGER, H. Über eine Erweiterung des Stieltjesschen Momentenproblems. *Math. Ann. 81*, 2-4 (1920), 235–319.

[44] HAMBURGER, H. Über eine Erweiterung des Stieltjesschen Momentenproblems. *Math. Ann. 82*, 3-4 (1921), 168–187.

[45] HAUSDORFF, F. Momentprobleme für ein endliches intervall. *Mathematische Zeitschrift 16*, 1 (1923), 220–248.

[46] HEAVISIDE, O. *Electromagnetic Theory*, vol. 1 of *Cambridge Library Collection - Technology*. Cambridge University Press, 2011.

[47] HOCHBERG, K. J., AND ORSINGHER, E. The arc-sine law and its analogs for processes governed by signed and complex measures. *Stochastic Process. Appl. 52*, 2 (1994), 273–292.

[48] IACUS, S. M., AND YOSHIDA, N. Estimation for the discretely observed telegraph process. *Teor. Ĭmovīr. Mat. Stat.*, 78 (2008), 32–42.

[49] IGNAT, L. I., AND STAN, D. Asymptotic behavior of solutions to fractional diffusion-convection equations. *J. Lond. Math. Soc. (2) 97*, 2 (2018), 258–281.

[50] IS, L., AMARAL, N., PLEROU, V., GOPIKRISHNAN, P., MEYER, M., AND STANLEY, H. The distribution of returns of stock prices. *International Journal of Theoretical and Applied Finance 3* (07 2000), 365–369.

[51] J. A. SHOHAT, J. D. T. *The Problem of Moments*, 4th ed. Mathematical Surveys and Monographs. American Mathematical Society, 1943.

[52] JACKSON, J. D. *Classical Electrodynamics Third Edition*, third ed. Wiley, Aug. 1998.

[53] JAIN, P., AND JAIN, S. On Young type inequalities for generalized convolution. *Proc. A. Razmadze Math. Inst. 164* (2014), 45–61.

[54] KAC, M. A stochastic model related to the telegrapher's equation. *Rocky Mountain J. Math. 4* (1974), 497–509. Reprinting of an article published in 1956.

[55] KADOMTSEV, B., AND PETVIASHVILI, V. On the stability of solitary waves in weakly dispersing media. *Dokl. Akad. Nauk SSSR 192* (1970), 753–756.

[56] KEMPPAINEN, J., SILJANDER, J., VERGARA, V., AND ZACHER, R. Decay estimates for time-fractional and other non-local in time subdiffusion equations in $\mathbb{R}^d$. *Math. Ann. 366*, 3-4 (2016), 941–979.

[57] KEMPPAINEN, J., SILJANDER, J., AND ZACHER, R. Representation of solutions and large-time behavior for fully nonlocal diffusion equations. *J. Differential Equations 263*, 1 (2017), 149–201.

[58] KEMPPAINEN, J., AND ZACHER, R. Long-time behavior of non-local in time fokker-planck equations via the entropy method. *Mathematical Models and Methods in Applied Sciences* (2019).

[59] KIM, Y.-J. A generalization of the moment problem to a complex measure space and an approximation technique using backward moments. *Discrete Contin. Dyn. Syst. 30*, 1 (2011), 187–207.

[60] KOCHUBEI, A. N. Distributed order calculus and equations of ultraslow diffusion. *J. Math. Anal. Appl. 340*, 1 (2008), 252–281.

[61] KOCHUBEI, A. N. General fractional calculus, evolution equations, and renewal processes. *Integral Equations Operator Theory 71*, 4 (2011), 583–600.

[62] KOCHUBEĬ, A. N. The Cauchy problem for evolution equations of fractional order. *Differentsial'nye Uravneniya 25*, 8 (1989), 1359–1368, 1468.

[63] KOCHUBEĬ, A. N. Diffusion of fractional order. *Differentsial'nye Uravneniya 26*, 4 (1990), 660–670, 733–734.

[64] KOLESNIK, A. D. Moment analysis of the telegraph random process. *Bul. Acad. Ştiinţe Repub. Mold. Mat.*, 1 (2012), 90–107.

[65] Kolesnik, A. D., and Ratanov, N. *Telegraph processes and option pricing.* SpringerBriefs in Statistics. Springer, Heidelberg, 2013.

[66] Koscielny-Bunde, E., Roman, H. E., Bunde, A., Havlin, S., and Schellnhuber, H. J. Long-range power-law correlations in local daily temperature fluctuations. *Philosophical Magazine Part B 77* (1998), 1331–1340.

[67] Krasnoschok, M., Pata, V., and Vasylyeva, N. Solvability of linear boundary value problems for subdiffusion equations with memory. *J. Integral Equations Appl. 30*, 3 (2018), 417–445.

[68] Laskin, N. Fractional quantum mechanics and Lévy path integrals. *Phys. Lett. A 268*, 4-6 (2000), 298–305.

[69] Laskin, N. Fractional Schrödinger equation. *Phys. Rev. E (3) 66*, 5 (2002), 056108, 7.

[70] Li, C.-G., Li, M., Piskarev, S., and Meerschaert, M. M. The fractional d'Alembert's formulas. *J. Funct. Anal. 277*, 12 (2019), 108279, 35.

[71] Lu, G. *The Peierls-Nabarro model of dislocations: a venerable theory and its current development.* In Handbook of Materials Modeling. Springer Netherlands, 2005.

[72] Mainardi, F. Fractional relaxation-oscillation and fractional diffusion-wave phenomena. *Chaos Solitons Fractals 7*, 9 (1996), 1461–1477.

[73] Mainardi, F. *Fractional calculus and waves in linear viscoelasticity.* Imperial College Press, London, 2010. An introduction to mathematical models.

[74] Mainardi, F., and Pironi, P. The fractional Langevin equation: Brownian motion revisited. In *Proceedings of the Third Meeting on Current Ideas in Mechanics and Related Fields (Segovia, 1995)* (1996), vol. 11, pp. 140–154.

[75] Mancinelli, R., Vergni, D., and Vulpiani, A. Front propagation in reactive systems with anomalous diffusion. *Phys. D 185*, 3-4 (2003), 175–195.

[76] Mascia, C. Stability analysis for linear heat conduction with memory kernels described by Gamma functions. *Discrete Contin. Dyn. Syst. 35*, 8 (2015), 3569–3584.

[77] Meerschaert, M. M., Nane, E., and Vellaisamy, P. Fractional Cauchy problems on bounded domains. *Ann. Probab. 37*, 3 (2009), 979–1007.

[78] Meerschaert, M. M., Nane, E., and Vellaisamy, P. The fractional Poisson process and the inverse stable subordinator. *Electron. J. Probab. 16* (2011), no. 59, 1600–1620.

[79] Meerschaert, M. M., and Scheffler, H.-P. Limit theorems for continuous-time random walks with infinite mean waiting times. *J. Appl. Probab. 41*, 3 (2004), 623–638.

[80] Meerschaert, M. M., and Scheffler, H.-P. Stochastic model for ultraslow diffusion. *Stochastic Process. Appl. 116*, 9 (2006), 1215–1235.

[81] Meerschaert, M. M., and Sikorskii, A. *Stochastic models for fractional calculus*, vol. 43 of *De Gruyter Studies in Mathematics.* Walter de Gruyter & Co., Berlin, 2012.

[82] Meerschaert, M. M., and Straka, P. Inverse stable subordinators. *Math. Model. Nat. Phenom. 8*, 2 (2013), 1–16.

[83] Metzler, R., and Klafter, J. The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Phys. Rep. 339*, 1 (2000), 77.

[84] Nikitin, Y., and Orsingher, E. On sojourn distributions of processes related to some higher-order heat-type equations. *J. Theoret. Probab. 13*, 4 (2000), 997–1012.

[85] Orsingher, E. Processes governed by signed measures connected with third-order "heat-type" equations. *Litovsk. Mat. Sb. 31*, 2 (1991), 323–336.

[86] Orsingher, E., and Beghin, L. Time-fractional telegraph equations and telegraph processes with Brownian time. *Probab. Theory Related Fields 128*, 1 (2004), 141–160.

[87] Orsingher, E., and Beghin, L. Fractional diffusion equations and processes with randomly varying time. *Ann. Probab. 37*, 1 (2009), 206–249.

[88] Orsingher, E., Ricciuti, C., and Toaldo, B. On semi-Markov processes and their Kolmogorov's integro-differential equations. *J. Funct. Anal. 275*, 4 (2018), 830–868.

[89] Orsingher, E., and Toaldo, B. Space-time fractional equations and the related stable processes at random time. *J. Theoret. Probab. 30*, 1 (2017), 1–26.

[90] Paul, C. R. *Analysis of Multiconductor Transmission Lines*, 2nd ed. Wiley-IEEE Press, 2007.

[91] Pishro-Nik, H. *Introduction to Probability, Stochastics and Random Processes.* Kappa Research, LLC, 2014.

[92] Ponce, R. A subordination principle for subdiffusion equations with memory. *J. Integral Equations Appl. 32*, 4 (2020), 479–493.

[93] Pozo, J. C., and Vergara, V. Fundamental solutions and decay of fully non-local problems. *Discrete & Continuous Dynamical Systems - A 39* (2019), 639–666.

[94] Pozo, J. C., and Vergara, V. A non-local in time telegraph equation. *Nonlinear Anal. 193* (2020), 111411, 22.

[95] Prüss, J. *Evolutionary integral equations and applications.* Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 1993. [2012] reprint of the 1993 edition.

[96] Reynolds, A., and Rhodes, C. The Lévy flight paradigm: Random search patterns and mechanisms. *Ecology 90* (2009), 877–887.

[97] Riesz, M. Sur le problème des moments et le théorème de parseval correspondant. *Scandinavian Actuarial Journal 1924*, 1 (1924), 54–74.

[98] Sadowski, J. Young's inequality for convolution and its applications in convex- and set-valued analysis. *J. Math. Anal. Appl. 421*, 2 (2015), 1274–1294.

[99] Samko, S. G., and Cardoso, R. P. Integral equations of the first kind of Sonine type. *Int. J. Math. Math. Sci.*, 57 (2003), 3609–3632.

[100] Samko, S. G., Kilbas, A. A., and Marichev, O. I. *Fractional integrals and derivatives.* Gordon and Breach Science Publishers, Yverdon, 1993. Theory and applications, Edited and with a foreword by S. M. Nikol prime skiui, Translated from the 1987 Russian original, Revised by the authors.

[101] Savov, M., and Toaldo, B. Semi-Markov processes, integro-differential equations and anomalous diffusion-aggregation. *Ann. Inst. Henri Poincaré Probab. Stat. 56*, 4 (2020), 2640–2671.

[102] Saxton, M. J. A biological interpretation of transient anomalous subdiffusion. i. qualitative model. *Biophysical journal 92*, 4 (2007), 1178–1191.

[103] Schilling, R. L., Song, R., and Vondraček, Z. *Bernstein functions*, vol. 37 of *De Gruyter Studies in Mathematics.* Walter de Gruyter & Co., Berlin, 2010. Theory and applications.

[104] Schneider, W. R., and Wyss, W. Fractional diffusion and wave equations. *J. Math. Phys. 30*, 1 (1989), 134–144.

[105] Serfaty, S., and Vázquez, J. L. A mean field equation as limit of nonlinear diffusions with fractional Laplacian operators. *Calc. Var. Partial Differential Equations 49*, 3-4 (2014), 1091–1120.

[106] Shlesinger, M. F. Origins and applications of the Montroll-Weiss continuous time random walk. *Eur. Phys. J. B 90*, 5 (2017), Paper No. 93, 5.

[107] Staelin, D. *Electromagnetics and Applications.* Massachusetts Institute of Technology: MIT OpenCourseWare 6.013, 2009.

[108] Stieltjes, T.-J. Recherches sur les fractions continues. *Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys. 8*, 4 (1894), J1–J122.

[109] Stieltjes, T.-J. Recherches sur les fractions continues [Suite et fin]. *Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys. 9*, 1 (1895), A5–A47.

[110] Toland, J. F. The Peierls-Nabarro and Benjamin-Ono equations. *J. Funct. Anal. 145*, 1 (1997), 136–150.

[111] Uchaikin, V. V. Montroll-Weiss problem, fractional equations, and stable distributions. *Internat. J. Theoret. Phys. 39*, 8 (2000), 2087–2105.

[112] Vergara, V. Asymptotic behaviour of the time-fractional telegraph equation. *J. Appl. Probab. 51*, 3 (2014), 890–893.

[113] Vergara, V., and Zacher, R. Optimal decay estimates for time-fractional and other nonlocal subdiffusion equations via energy methods. *SIAM J. Math. Anal. 47*, 1 (2015), 210–239.

[114] Viswanathan, G. M., Afanasyev, V., Buldyrev, S. V., Murphy, E. J., Prince, P. A., and Stanley, H. Lévy flight search patterns of wandering albatrosses. *Nature 381* (1996), 413–415.

[115] Weil, A. *L'intégration dans les groupes topologiques et ses applications. Deuxième édition*, 2nd ed. Hermann, 1979.

[116] Zaslavsky, G. M. Anomalous transport in systems with Hamiltonian chaos. In *Transport, chaos and plasma physics (Marseille, 1993)*. World Sci. Publ., River Edge, NJ, 1994, pp. 2–29.