Representations of groups of automorphisms on compact Riemann surfaces

Pablo Alvarado Seguel
Universidad de Chile

I Escuela de Postgrado en Matemáticas
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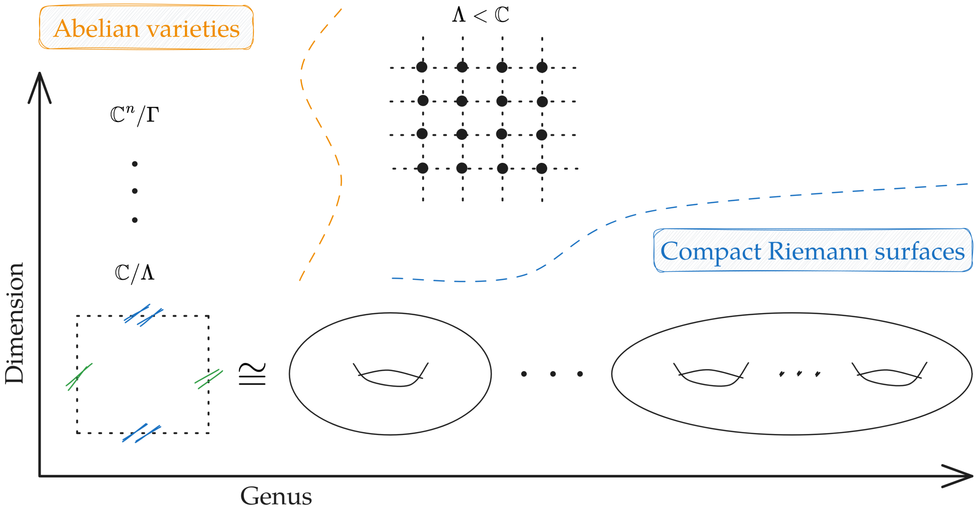
Preliminaries

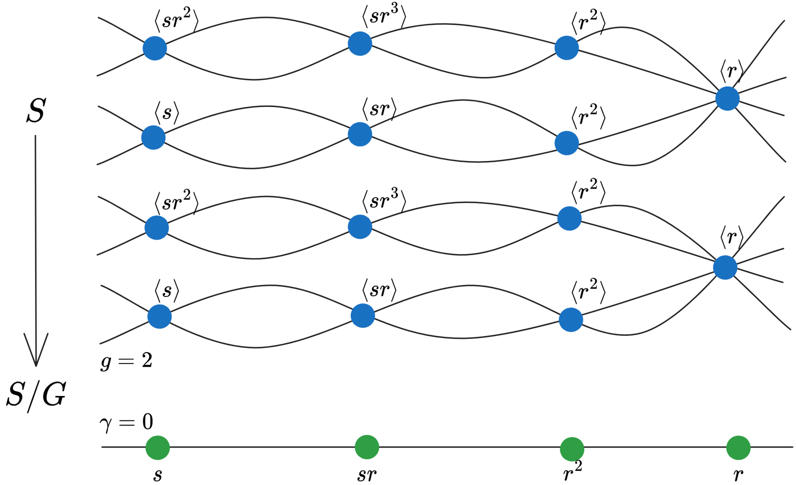
Results: Dihedral actions

- Geometric signature and analytic representation
- Existence theorems
- Group algebra decomposition

Geometric structures



Actions on compact Riemann surfaces



$G \cong D_4 = \langle r, s : r^4, s^2, (sr)^2 \rangle$

Signature
$(0; 2, 2, 2, 4)$

Geometric signature
$(0; \langle s \rangle, \langle sr \rangle, \langle r^2 \rangle, \langle r \rangle)$

Generating vector
(s, sr, r^2, r)

Actions on compact Riemann surfaces

Theorem 1

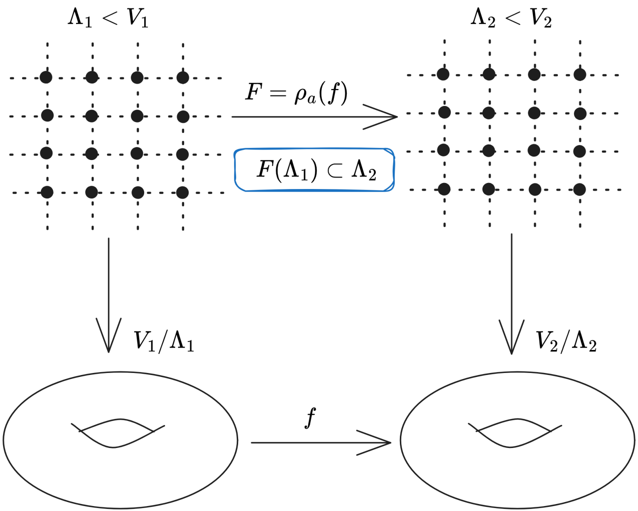
Let S be a compact Riemann surface of genus $g \geq 2$. A (finite) group G acts on S with signature $(\gamma; m_1, \ldots, m_v)$ if and only if the following conditions are satisfied:

1. *(Riemann-Hurwitz)*

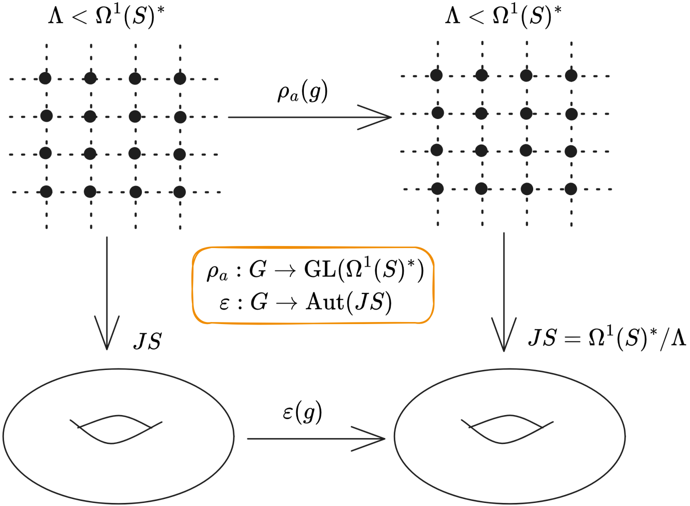
$$g = 1 + |G|(\gamma - 1) + \frac{|G|}{2} \sum_{j=1}^{v} \left(1 - \frac{1}{m_j}\right).$$

2. *The group G has a generating vector of type $(\gamma; m_1, \ldots, m_v)$.*

Analytic and rational representation



Analytic representation of a group action



Analytic representation of a group action

- $\rho_r \otimes 1 \cong \rho_a \oplus \overline{\rho_a}$
- Geometric signature $\rightarrow \rho_r$.
- **Chevalley-Weil formula**: Generating vector $\rightarrow \rho_a$.

$$\langle \rho_a, V \rangle = d_V(\gamma - 1) + \sum_{j=1}^{v} \mathcal{N}(V, c_j)$$

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Preliminaries

Results: Dihedral actions

Geometric signature and analytic representation

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Group algebra decomposition

Notation

Consider the dihedral group

$$D_n = \langle s, r : r^n, s^2, (sr)^2 \rangle.$$

If n is even then D_n has four $\mathbb{C}$-representations of degree one:

$$\psi_1 : r \mapsto 1,\ s \mapsto 1; \qquad \psi_2 : r \mapsto 1,\ s \mapsto -1;$$
$$\psi_3 : r \mapsto -1,\ s \mapsto 1; \qquad \psi_4 : r \mapsto -1,\ s \mapsto -1;$$

and $(n-2)/2$ irreducible $\mathbb{C}$-representations of degree two:

$$\rho^h : r \mapsto \mathrm{diag}(\omega^h, \overline{\omega}^h),\ s \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right),$$

where $\omega = e^{2\pi i/n}$ and $1 \leq h \leq (n-2)/2$.

Notation

Definition 2

Set $G = D_n$. Consider a G-action with geometric signature

$$(\gamma; \langle s\rangle^a, \langle sr\rangle^b, C_1, \ldots, C_v)$$

where $C_j = \langle r^{n/m_j}\rangle$ is a cyclic group of order $m_j \geq 2$. Then the *signature function* is

$$\Psi_\theta : \mathbb{Z}_+ \to \mathbb{Z}, \quad \Psi_\theta(q) = \#\{1 \leq j \leq v : m_j = q\}.$$

Definition 3

Let $\Psi : \mathbb{Z}_+ \to \mathbb{Z}$ be a function. The *divisor transform* of Ψ is the function

$$\widehat{\Psi} : \mathbb{Z}_+ \to \mathbb{Z}, \quad \widehat{\Psi}(n) = \sum_{q|n} \Psi(q).$$

Analytic representation formula: D_n with n even

Theorem 4

Set $G = D_n$ with $n \geq 2$ an even integer, and let S be a compact Riemann surface of genus $g \geq 2$. If S has a G-action geometric signature $(\gamma; \langle s \rangle^a, \langle sr \rangle^b, C_1, \ldots, C_v)$, then

$$\rho_a \cong \gamma\psi_1 \oplus \mu_2\psi_2 \oplus \mu_3\psi_3 \oplus \mu_4\psi_4 \oplus \bigoplus_{h=1}^{(n-2)/2} \nu_h \rho^h,$$

where

$$\mu_2 = \langle \rho_a, \psi_2 \rangle = \gamma - 1 + \tfrac{1}{2}[a + b],$$
$$\mu_3 = \langle \rho_a, \psi_3 \rangle = \gamma - 1 + \tfrac{1}{2}[b + \widehat{\Psi}_\theta(n) - \widehat{\Psi}_\theta(\tfrac{n}{2})],$$
$$\mu_4 = \langle \rho_a, \psi_4 \rangle = \gamma - 1 + \tfrac{1}{2}[a + \widehat{\Psi}_\theta(n) - \widehat{\Psi}_\theta(\tfrac{n}{2})],$$
$$\nu_h = \langle \rho_a, \rho^h \rangle = 2(\gamma - 1) + \tfrac{1}{2}[a + b] + \widehat{\Psi}_\theta(n) - \widehat{\Psi}_\theta(h), \quad 1 \leq h \leq (n-2)/2.$$

Example: D_p with p prime

Set $G = D_p$ with $p \geq 3$ a prime number. If G acts with signature $(\gamma; 2^t, p^l)$, then

$$\rho_a \cong \gamma\psi_1 \oplus [\gamma - 1 + \tfrac{1}{2}t]\psi_2 \oplus [2(\gamma - 1) + \tfrac{1}{2}t + l]\left(\bigoplus_{h=1}^{(p-1)/2} \rho^h\right).$$

The dihedral group D_7 acts on genus $g = 12$ with signature $(0; 2, 2, 7, 7, 7)$. Its analytical representation is

$$\rho_a \cong 2(\rho^1 \oplus \rho^2 \oplus \rho^3).$$

Bijective correspondence

Theorem 5

Set $G = D_n$. There is a bijection between the geometric signatures and the analytic representations of the G-actions.

Existence of actions with a given signature

Theorem 6 (Bujalance et al, 2003)

Set $G = D_n$ with n an even integer. Assume that $\gamma = 0$. Then the necessary and sufficient conditions for the existence of a compact Riemann surface $(g \geq 2)$ with a G-action of signature $(\gamma; 2^t, m_1, \ldots, m_v)$ are:

1. $t \geq 2$;
2. *if* $t = 2$ *then* $\mathrm{lcm}(m_1, \ldots, m_v) = n$;
3. *if* $t = 3$ *then* $\mathrm{lcm}(2, m_1, \ldots, m_v) = n$.

Existence of actions with a given geometric signature

Theorem 7

Set $G = D_n$ with n an even integer. Assume that $\gamma = 0$. Then the necessary and sufficient conditions for the existence of a compact Riemann surface $(g \geq 2)$ with a G-action of geometric signature

$$(\gamma; \langle s \rangle^a, \langle sr \rangle^b, C_1, \dots, C_v)$$

are:

1. *a and b have the same parity as $A := \#\{1 \leq j \leq v : n/m_j$ is odd$\}$;*
2. *$a + b \geq 2$ is even;*
3. *if $a + b = 2$ then $\mathrm{lcm}(m_1, \dots, m_v) = n$;*
4. *if $a + b > 2$ and $a = 0$ or $b = 0$ then $A > 0$.*

Analytic representation criteria

Theorem 8

Let V be a $\mathbb{C}$-representation of $G = D_n$ with n even. Assume that $\langle V, \psi_1 \rangle = \gamma = 0$. Then the necessary and sufficient conditions for V to be equivalent to the analytic representation ρ_a of a G-action are:

1. $\langle V, \psi_2 \rangle \geq |\langle V, \psi_3 \rangle - \langle V, \psi_4 \rangle| - 1$ *and* $\widetilde{\Phi}_V(q) \geq 0$ *for each divisor* $q > 1$ *of* n*;*
2. $\langle V, \rho^h \rangle = \langle V, \rho^{(n,h)} \rangle$ *for* $1 \leq h \leq (n-2)/2$*;*
3. *if* $\langle V, \psi_2 \rangle = 0$ *then* $\operatorname{lcm}(m_1, \ldots, m_v) = n$*;*
4. *if* $\langle V, \psi_2 \rangle \geq 1$ *and* $|\langle V, \psi_3 \rangle - \langle V, \psi_4 \rangle| = \langle V, \psi_2 \rangle + 1$ *then* $A > 0$*.*

Decomposition theorems

Set $\mathrm{Irr}_{\mathbb{Q}}(G) = \{W_1, \ldots, W_v\}$.

Theorem 9

Let A be an abelian variety with a G-action. There are abelian subvarieties $B_1, \ldots, B_v$ of A and a G-equivariant isogeny

$$A \sim B_1^{n_1} \times \cdots \times B_v^{n_v},$$

where G acts on $B_i^{n_j}$ via the representation W_j.

For $j = 2, \ldots, v$, the dimension of the group algebra component B_j of JS is

$$\dim B_j = \frac{1}{2} k_{V_j} \langle \rho_r \otimes 1, V_j \rangle,$$

where $V_j \in \mathrm{Irr}_{\mathbb{C}}(G)$ is Galois associated to W_j.

Ekedahl-Serre problem

Theorem 10

Set $G = D_n$. The group algebra decomposition of JS respect to a G-action is never affordable by elliptic curves, unless $n \in \{3, 4, 6\}$.

- An exhaustive list of geometric signatures is found for when JS is affordable by elliptic curves.
- An exhaustive list of geometric signatures is found for when JS is d-affordable.

References

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