ON THE ASYMPTOTIC OF A LAZY REINFORCED RANDOM WALK

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(A) Introduction

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INTRODUCTION

The one-dimensional simple random walk is defined by an independent identically distributed sequence $\{X_1, X_2, \ldots\}$ where $X_i \in \{-1, +1\}$. The main interest is the position of the walker at time n, given by

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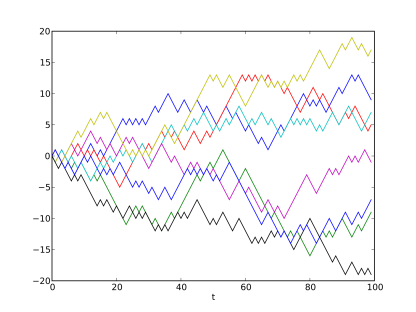

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In other words, at step $n + 1$, it chooses $t \in \{1, ...n\}$ uniformly at random. Then

$$X_{n+1} = \begin{cases} X_t & \text{with probability } p \\ -X_t & \text{with probability } 1 - p \end{cases} \quad (1)$$

The ERW shows a transition from diffusive to super-diffusive behaviours for S_n, with critical $p_c = \frac{3}{4}$. That is, the mean squared displacement is a linear function of time in the diffusive case ($p < p_c$), but is given by a power law in the super-diffusive regime ($p > p_c$)

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Formulation of the model

We are interested in the formulation of an ERW with delays as given by Gut and Stadtmüller (2019). Let the first step given by

$$X_1 = \begin{cases} +1 & \text{, with probability } p, \\ -1 & \text{, with probability } q, \\ 0 & \text{, with probability } r. \end{cases} \tag{2}$$

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The next steps are performed by the rule

$$X_{n+1} = \begin{cases} X_t & \text{, with probability } p, \\ -X_t & \text{, with probability } q, \\ 0 & \text{, with probability } r, \end{cases} \tag{3}$$

where t is uniformly chosen from $\{1, \ldots, n\}$.

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$$\alpha_n = \begin{cases} 1, & \text{with probability } p, \\ -1, & \text{with probability } q, \\ 0, & \text{with probability } r, \end{cases}$$

with $p + q + r = 1$, such that $X_1 = \alpha_1$ and for each $n \geq 2$ we set

$$X_n = Y_n \alpha_n X_{U_n} + (1 - Y_n)\,\alpha_n, \tag{4}$$

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where Y_n posses the Bernoulli distribution with parameter $\theta \in [0, 1)$. U_n is a discrete uniform random variable on $\{1, 2, \ldots, n-1\}$. Moreover, α_n and U_n are independent and Y_n is independent of the RW's past.

MAIN RESULTS

We use the following notation

$$\alpha = (\mathrm{p} - \mathrm{q}) \cdot \theta, \ \omega = (\mathrm{p} - \mathrm{q})(1 - \theta), \ \tau = (1 - \theta)(\mathrm{p} + \mathrm{q}),$$

$$\gamma = (\mathrm{p} + \mathrm{q}) \cdot \theta \text{ and } \sigma^2 = \frac{\tau}{1 - \gamma} - \left(\frac{\omega}{1 - \alpha} \right)^2 . \tag{5}$$

Theorem (G-N et. al, 2024)

Let the RW given by (4), for all $\alpha \in [0, 1)$

$$\lim_{n\to\infty} \frac{S_n - \mathbb{E}(S_n)}{n} = 0 \quad \text{a.s} \tag{6}$$

and

$$\lim_{n\to\infty} \frac{S_n}{n} = \frac{\omega}{1-\alpha} \quad \text{a.s.} \tag{7}$$

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If $\alpha < 1/2$, we have the distributional convergence in $D([0,\infty[)$,

$$\left(\sqrt{n}\Big(\frac{S_{\lfloor nt \rfloor}}{\lfloor nt \rfloor} - \frac{\omega}{1-\alpha}\Big), t \geq 0\right) \Longrightarrow \big(W_t, t \geq 0\big) \tag{8}$$

where $\big(W_t, t \geq 0\big)$ is a real-valued centered Gaussian process starting at the origin with covariance given, for all $0 < s \leq t$, by $\mathbb{E}[W_s W_t] = \frac{\sigma^2}{(1-2\alpha)t}\Big(\frac{t}{s}\Big)^{\alpha}$.

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$$\sqrt{n}\Big(\frac{S_n}{n} - \frac{\omega}{1-\alpha}\Big) \xrightarrow{d} N\Big(0, \frac{\sigma^2}{1-2\alpha}\Big). \tag{9}$$

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If $\alpha < 1/2$, then

$$\limsup_{n\to\infty} \pm \left(\frac{n}{2\log\log n}\right)\left(\frac{S_n}{n} - \frac{\omega}{1-\alpha}\right)^2 = \frac{\sigma^2}{1-2\alpha} \qquad \text{a.s.}$$

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Theorem (G-N et. al, 2024)

If $\alpha < 1/2$ then we have the following almost sure convergence of empirical measures

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbb{I}_{\left\{\sqrt{k}\left(\frac{S_k}{k}-\frac{\omega}{1-\alpha}\right)\le x\right\}} \xrightarrow{n\to\infty} F_Z(x) \quad \text{a.s} \tag{10}$$

where F_Z is the cumulative distribution function of $Z \sim N(0, \sigma^2/(1-2\alpha))$.

Theorem (G-N et. al, 2024)

If $\alpha = 1/2$, we have the distributional convergence in $D([0,\infty[)$,

$$\left(\sqrt{\frac{n^t}{\log n}}\Big(\frac{S_{\lfloor n^t \rfloor}}{\lfloor n^t \rfloor} - 2\omega\Big), t \geq 0\right) \Longrightarrow \left(\frac{\tau}{1-\gamma} - 4\omega^2 B_t, t \geq 0\right) \qquad (11)$$

where $(B_t, t \geq 0)$ is a standard Brownian motion. In particular, we have the asymptotic normality

$$\sqrt{\frac{n}{\log n}}\left(\frac{S_n}{n} - 2\omega\right) \xrightarrow{d} N\left(0, \frac{\tau}{1-\gamma} - 4\omega^2\right). \qquad (12)$$

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We have the almost sure convergence

$$\left(n^{1-\alpha}\Big(\frac{S_{\lfloor nt\rfloor}}{\lfloor nt\rfloor} - \frac{\omega}{1-\alpha}\Big), t > 0\right) \longrightarrow \Big(\frac{1}{t^{1-a}}L, t > 0\Big) \tag{14}$$

where L is a non-degenerated random variable such that

$$\mathbb{E}[L] = \frac{\beta(1-\alpha) - \omega}{\Gamma(\alpha+1)(1-\alpha)} \tag{15}$$

where $\beta := p - q$, and

$$\mathbb{E}[L^2] = \frac{\nabla}{\Gamma(2\alpha+1)} + 2\omega\left(\frac{1}{(1-\alpha)\Gamma(\alpha)}\right)^2 \tag{16}$$

where $\nabla := p + q + \frac{\tau}{(1-\gamma)(2\alpha-1)} - \frac{2\alpha\omega^2}{(2\alpha-1)(\alpha-1)^2} + 4\left[\frac{\omega\alpha(\beta-1)}{(\alpha-1)^2}\right] + \frac{r\gamma^2}{2\alpha-\gamma}$.

Theorem (G-N et. al, 2024)

If $\alpha > 1/2$, then

$$\sqrt{n^{2\alpha-1}}\left(n^{1-\alpha}\left(\frac{S_n}{n} - \frac{\omega}{1-\alpha}\right) - L\right) \xrightarrow{d} N\left(0, \frac{\sigma^2}{2\alpha-1}\right) \text{ as } n \to \infty \quad (17)$$

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and

$$\limsup_{n\to\infty} \pm \frac{\sqrt{n^{2\alpha-1}}\left(n^{1-\alpha}\left(\frac{S_n}{n}-\frac{\omega}{1-\alpha}\right)-L\right)}{\sqrt{\log\log n}} = \sqrt{\frac{2\sigma^2}{2\alpha-1}} \text{ a.s} \tag{18}$$

SKETCH OF THE PROOFS

We base the asymptotic analysis of the RW on the sequence (M_n), given by $M_0 = 0$ and for $n \geq 1$ by

$$M_n = a_n S_n - \omega A_n, \tag{19}$$

where; on the one hand, the sequence (a_n) is given by $a_1 = 1$ and for $n \geq 2$ as

$$a_n = \prod_{k=1}^{n-1} \gamma_k^{-1} = \frac{\Gamma(n)\Gamma(\alpha+1)}{\Gamma(n+\alpha)} \sim \frac{\Gamma(1+\alpha)}{n^{\alpha}}, \tag{20}$$

where Γ stands for the Euler gamma function, and; on the other hand, sequence (A_n) is given by $A_0 = 0$ and for $n \geq 1$ as

$$A_n = \sum_{k=1}^{n} a_n. \tag{21}$$

Additionally, we observe that from (20) that almost surely

$$\begin{aligned}\mathbb{E}[M_{n+1}|\mathcal{F}_n] &= a_{n+1}(\gamma_n S_n + \omega) - \omega A_{n+1} \\ &= a_n S_n - \omega A_n = M_n.\end{aligned}$$

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Thus, (M_n) is a discrete time martingale with respect to the filtration $(\mathcal{F}_n)$.

In this sense, the asymptotic behaviour of the model is strictly related with the sum:

$$v_n = \sum_{k=1}^{n} \frac{1}{a_k^2} \tag{22}$$

Now, note that by Stirling formula for the gamma function

$$a_n \sim \frac{n^\alpha}{\Gamma(\alpha+1)} \quad \text{as} \quad n \to \infty \tag{23}$$

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Therefore, in the **diffusive region**, where $0 \leq \alpha < 1/2$, we have:

$$v_n = \sum_{k=1}^{n} \left(\frac{\Gamma(k)\Gamma(\alpha+1)}{\Gamma(k+\alpha)} \right)^2 \sim (\Gamma(\alpha+1))^2 \sum_{k=1}^{n} \frac{1}{k^{2\alpha}} \tag{24}$$

as $n \to \infty$. Then, by the p-series we know that

$$\lim_{n\to\infty} \frac{v_n}{n^{1-2\alpha}} = \frac{(\Gamma(\alpha+1))^2}{1-2\alpha} \tag{25}$$

In the **critical region**, where $\alpha = 1/2$, we have

$$v_n \sim (\Gamma(3/2))^2 \sum_{k=1}^{n} \frac{1}{k} \tag{26}$$

Then v_n diverges with velocity $\log n$ and we obtain

$$\lim_{n\to\infty} \frac{v_n}{\log n} = \frac{\pi}{4} \tag{27}$$

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Finally, in the **superdiffusive region**, if $1/2 < \alpha \leq 1$,

$$\lim_{n\to\infty} v_n = \sum_{k=0}^{\infty} \left(\frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} \right)^2 = {}_3F_2 \left(\begin{array}{c} 1,1,1 \\ (\alpha+1),(\alpha+1) \end{array} \middle| \; 1 \right) \tag{28}$$

where ${}_3F_2$ is the (finite) hypergeometric generalized function.

Most of the asymptotic analysis will be conducted by the increasing process of martingale (M_n); the predictable quadratic variation $\langle M\rangle_n$ given, for all $n \geq 1$, by

$$\begin{aligned} \langle M\rangle_n &= \sum_{k=1}^{n} \mathbb{E}[\Delta M_k^2|\mathcal{F}_{k-1}] = \mathbb{E}\left[\xi_1^2|\mathcal{F}_0\right] + \sum_{k=1}^{n-1} a_{k+1}^2 \mathbb{E}\left[\xi_{k+1}^2|\mathcal{F}_k\right] \\ &= 1 - 2\omega(2\beta - 1) + \omega^2 + \gamma \sum_{k=1}^{n-1} a_{k+1}^2 \frac{Z_k}{k} + (\tau - \omega^2) v_n \\ &- 2\omega\alpha \sum_{k=1}^{n-1} a_{k+1}^2 \frac{S_k}{k} - \alpha^2 \sum_{k=1}^{n-1} a_{k+1}^2 \frac{S_k^2}{k^2} \qquad (29) \end{aligned}$$

Lemma

The martingale (M_n) can be written in the additive form

$$M_n = \sum_{k=1}^{n-1} \Delta M_k = \sum_{k=1}^{n-1} (M_k - M_{k-1}) = \sum_{k=1}^{n-1} \frac{X_k - \mathbb{E}(X_k | \mathcal{F}_{k-1})}{a_k} \tag{30}$$

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Let a_n defined in (20), then $\sum_{l=1}^{n-1} \frac{1}{a_{l+1}} \sim \frac{\Gamma(\alpha+1)n^{1-\alpha}}{(1-\alpha)}$.

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Lemma

The series $v_n = \sum \frac{1}{a_n^2}$ converges, if and only if, $\alpha > \frac{1}{2}$

Lemma

Let $\Delta M_n = M_n - M_{n-1}$, assume for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \frac{1}{v_n} \mathbb{E}\left[|\Delta M_n|^2 \mathbb{I}_{\{|\Delta M_n| \geq \varepsilon \sqrt{v_n}\}} | \mathcal{F}_{n-1}\right] < \infty \ \text{ a.s} \tag{31}$$

and for some $a > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{v_n^a} \mathbb{E}\left[|\Delta M_n|^{2a} \mathbb{I}_{\{|\Delta M_n| \leq \sqrt{v_n}\}} | \mathcal{F}_{n-1}\right] < \infty \ \text{ a.s} \tag{32}$$

Then, (M_n) satisfies that

$$\frac{1}{\log v_n} \sum_{k=1}^{n} \left(\frac{v_k - v_{k-1}}{v_k} \right) \delta_{M_k / \sqrt{v_{k-1}}} \Rightarrow G \quad \text{a.s} \tag{33}$$

where G stands for the $N(0, \sigma^2)$ distribution.

Proof of almost sure central limit theorem

The proof is essentially based on previous Lemma. Hence, we have that

$$\sum_{k=1}^{\infty} \frac{1}{v_k} \mathbb{E}\left[|\Delta M_k|^2 \mathbb{I}_{|\Delta M_k| \geq \varepsilon \sqrt{v_k}} | \mathcal{F}_{k-1}\right] \leq \frac{1}{\varepsilon^2} \sum_{k=1}^{\infty} \frac{1}{v_k^2} \mathbb{E}\left[|\Delta M_k|^4 | \mathcal{F}_{k-1}\right]$$

$$\leq \sup_{k \geq 1} \mathbb{E}\left[\xi_k^4 | \mathcal{F}_{k-1}\right] \frac{1}{\varepsilon^2} \sum_{k=1}^{\infty} \frac{a_k^4}{v_k^2} \leq \frac{16}{\varepsilon^2} \sum_{k=1}^{\infty} \frac{a_k^4}{v_k^2} \sim \frac{16}{\varepsilon^2} \sum_{k=1}^{\infty} \frac{(1-2\alpha)^2}{k^2} < \infty$$

Where, last step is due to (20). Therefore (31) holds. To prove the validity of (32) we follow analogous steps with $a = 2$. Then

$$\frac{1}{\log v_n} \sum_{k=1}^{n} \left(\frac{v_k - v_{k-1}}{v_k} \right) \delta_{M_k / \sqrt{v_{k-1}}} \Rightarrow G \quad \text{a.s} \qquad (34)$$

By recalling that, $f_k \sim \frac{1-2\alpha}{k}$, $\log v_n \sim (1-2\alpha)\log n$ and $\frac{M_k}{\sqrt{v_{k-1}}} \sim \sqrt{\frac{1-2\alpha}{k}}\left(S_k - k\frac{\omega}{1-\alpha}\right)$, we conclude that

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\delta_{\sqrt{k}\left(\frac{S_k}{k}-\frac{\omega}{1-\alpha}\right)} \Rightarrow G^* \quad \text{a.s} \tag{35}$$

where $G^* \sim N(0, \sigma^2/(1-2\alpha))$ is the re-scaled version of $G \sim N(0, \sigma^2)$.

Proof of functional central limit theorem

Note that, using (29) and Toeplitz lemma, we have

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n^{1-2\alpha}} \langle M \rangle_n &= \frac{\Gamma(\alpha+1)^2}{1-2\alpha} \left(\frac{\gamma\tau}{1-\gamma} + (\tau - \omega^2) - \frac{2\omega^2\alpha}{1-\alpha} - \left(\frac{\omega\alpha}{1-\alpha} \right)^2 \right) \\ &= \sigma^2 \frac{\Gamma^2(\alpha+1)}{1-2\alpha} \qquad \text{a.s.}\end{aligned}$$

Then, we apply the functional central limit theorem for martingales. That is, consider the martingale difference array $D_{n,k} = \frac{1}{\sqrt{n^{1-2\alpha}}}(\Delta M_k)$, which satisfies

$$\lim_{n\to\infty} \frac{1}{n^{1-2\alpha}} \langle M \rangle_{\lfloor nt \rfloor} = \sigma^2 \frac{\Gamma^2(\alpha+1)}{1-2\alpha} t^{1-2\alpha} \qquad \text{a.s.} \tag{36}$$

In addition, we need to prove the Lindeberg's condition.

$$
\begin{aligned}
\frac{1}{n^{1-2\alpha}}\sum_{k=1}^{n}\mathbb{E}[\Delta M_k^2 \mathbb{I}_{\{|\Delta M_k|>\varepsilon\sqrt{n^{1-2\alpha}}\}}|\mathcal{F}_{k-1}] &\leq \frac{1}{n^{2(1-2\alpha)}\varepsilon^2}\sum_{k=1}^{n}\mathbb{E}[\Delta M_k^4|\mathcal{F}_{k-1}] \\
\leq \frac{1}{n^{2(1-2\alpha)}\varepsilon^2}\sum_{k=1}^{n} a_k^4 \mathbb{E}[\xi_k^4|\mathcal{F}_{k-1}] &\leq \frac{16}{n^{2(1-2\alpha)}\varepsilon^2}\sum_{k=1}^{n} a_k^4,
\end{aligned}
$$

Then, thanks to (20), we have that, as $n \to \infty$

$$
\frac{n^2 a_n^4}{v_n^2} \to (1-2\alpha)^2,
$$

which implies that $\frac{1}{n^{1-4\alpha}}\sum_{k=1}^{n} a_k^4$ converges to $\frac{(1-2\alpha)^2\ell^2}{1-4\alpha}$.

Therefore,

$$\frac{1}{n^{1-2\alpha}} \sum_{k=1}^{n} \mathbb{E}[\Delta M_k^2 \mathbb{I}_{\{|\Delta M_k| > \varepsilon \sqrt{n^{1-2\alpha}}\}} | \mathcal{F}_{k-1}] \to 0 \text{ as } n \to \infty \text{ in probability,}$$

which allows us to conclude that for all $t \geq 0$ and for any $\varepsilon > 0$,

$$\frac{1}{n^{1-2\alpha}} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}[\Delta M_k^2 \mathbb{I}_{\{|\Delta M_k| > \varepsilon \sqrt{n^{1-2\alpha}}\}} | \mathcal{F}_{k-1}] \to 0, \tag{37}$$

as $n \to \infty$ in probability.

By noticing that $\lim_{n\to\infty} \frac{\lfloor nt \rfloor a_{\lfloor nt \rfloor}}{n^{1-2\alpha}} = t^{1-\alpha}\Gamma(\alpha+1)$ and that (20) implies that

$$\frac{M_{\lfloor nt \rfloor}}{\sqrt{n^{1-2\alpha}}} = \frac{\lfloor nt \rfloor a_{\lfloor nt \rfloor}}{\sqrt{n^{1-2\alpha}}} \left(\frac{S_{\lfloor nt \rfloor}}{\lfloor nt \rfloor} - \frac{\omega}{1-\alpha} \right) + \frac{\omega\alpha}{(1-\alpha)\sqrt{n^{1-2\alpha}}} \qquad \text{a.s.,} \tag{38}$$

we conclude that

$$\left(\sqrt{n} \left(\frac{S_{\lfloor nt \rfloor}}{\lfloor nt \rfloor} - \frac{\omega}{1-\alpha} \right), t \geq 0 \right) \Longrightarrow \left(W_t, t \geq 0 \right),$$

where $W_t = B_t/(t^{1-\alpha}\Gamma(\alpha+1))$, which completes the proof of the theorem.

Calculations in the superdiffusive regime

In this case, the second moment of the position is calculated recursively. That is, In this case, the second moment of the position is calculated recursively. That is,

$$\mathbb{E}[S_n^2] = \frac{\Gamma(n+2\alpha)}{\Gamma(n)\Gamma(2\alpha+1)} \left(p + q + \Gamma(2\alpha+1) \sum_{k=1}^{n-1} h_k \frac{\Gamma(k+1)}{\Gamma(k+1+2\alpha)} \right), \tag{39}$$

where

$$\begin{aligned} h_k \frac{\Gamma(k+1)}{\Gamma(k+1+2\alpha)} &= \frac{\tau}{1-\gamma} \frac{\Gamma(k+1)}{\Gamma(k+1+2\alpha)} + \frac{2\omega^2}{1-\alpha} \frac{k\Gamma(k+1)}{\Gamma(k+1+2\alpha)} \\ &- t_1 \frac{\Gamma(k+1)}{a_k \Gamma(k+1+2\alpha)} + \gamma t_2 \frac{\Gamma(k+1)}{k b_k \Gamma(k+1+2\alpha)} \end{aligned}$$

MULTIDIMENSIONAL WALKS WITH TENDENCY

We define a discrete-time evolution $(X_i)_{i\geq 1}$. The n-step denotes an opinion (movement), given by $X_n \in E = \{1, 2, \ldots, K\}$ the set of choices.

We define a discrete-time evolution $(X_i)_{i \geq 1}$. The n-step denotes an opinion (movement), given by $X_n \in E = \{1, 2, \ldots, K\}$ the set of choices. In the context of a random walk, we have $K = 2d$ or $2d + 1$ with laziness, then, we denote the set of directions by

$$E_d = \begin{cases} (e_1, -e_1, \ldots, e_d, -e_d) & , \text{ if K is even,} \\ (e_1, -e_1, \ldots, e_d, -e_d, \mathbf{0}) & , \text{ if K is odd,} \end{cases}$$

where $(e_1, \ldots, e_d)$ is the canonical basis of the Euclidean space $\mathbb{R}^d$, and $\mathbf{0}$ denotes not movement.

Let $S_n = \sum_{i=1}^{n} X_i$ the d-dimensional position of the walker at time n.
The $(n+1)$-step is obtained by flipping a coin with probability θ, denoted Y_n and then:

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- If $Y_n = 1$, we chose uniformly at random $t \in \{1, 2, \ldots, n\}$, then X_{n+1} is equal to X_t with probability p. Otherwise, X_{n+1} follows any other direction with uniform probability $\frac{1-p}{K-1}$.

Let $S_n = \sum_{i=1}^{n} X_i$ the d-dimensional position of the walker at time n. The $(n+1)$-step is obtained by flipping a coin with probability θ, denoted Y_n and then:

- If $Y_n = 1$, we chose uniformly at random $t \in \{1, 2, \dots, n\}$, then X_{n+1} is equal to X_t with probability p. Otherwise, X_{n+1} follows any other direction with uniform probability $\frac{1-p}{K-1}$.
- If $Y_n = 0$, then $X_{n+1} = e_1$ with probability p or any other direction with uniform probability $\frac{1-p}{K-1}$.

Note that, if $\theta = 1$ we obtain an elephant-type dynamics. In case $\theta = 0$, the tendency with intensity p is given by direction e_1, such tendency is effective if $p > 1/K$.

Main results

Theorem (G-N, 2020)

Let $(S_n)_{n\in\mathbb{N}}$ the position of the walker, we get the following almost-surely convergence

$$\lim_{n\to\infty} \frac{S_n}{n} = \frac{(1-\theta)(Kp-1)}{K-1+\theta(1-Kp)} (1, 0, \ldots, 0)^T .$$

Theorem (G-N, 2020)

If $p < \frac{K+2\theta-1}{2\theta K}$ then, for $n \to \infty$, in $D[0,\infty)$

$$\frac{1}{\sqrt{n}}\left[S_{\lfloor tn \rfloor} - \frac{tn(1-\theta)(Kp-1)}{K-1+\theta(1-Kp)}(1,0,\dots,0)^{T}\right] \xrightarrow{d} W_t,$$

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where W_t is a continuous d-dimensional Gaussian process with $W_0 = (0,\dots,0)^T$, $\mathbb{E}(W_t) = (0,\dots,0)^T$ and, for $0 < s \leq t$,

$$\mathbb{E}(W_s W_t^T) = s\left(\frac{t}{s}\right)^{\frac{\theta(Kp-1)}{(K-1)}} \omega \begin{pmatrix} (K+1)\alpha+\beta+p-1 & 0 & \dots & 0 \\ 0 & 2\beta & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 2\beta \end{pmatrix}$$

where $\omega = \dfrac{(K-1)(1-p)}{\beta^2(K-1+2\theta(1-Kp))}$, $\alpha = (K-1)p + \theta(1-Kp)$ and $\beta = K-1+\theta(1-Kp)$.

Theorem (G-N, 2020)

If $p = \frac{K+2\theta-1}{2\theta K}$ then, for $n \to \infty$, in $D[0, \infty)$

$$\frac{1}{\sqrt{n^t \log(n)}} \left[S_{\lfloor n^t \rfloor} - n^t \frac{K(2p-1)-1}{K-1} (1, 0, \ldots, 0)^T \right] \xrightarrow{d} W_t,$$

where W_t as above and for $0 < s \leq t$,

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where W_t as above and for $0 < s \leq t$,

$$\mathbb{E}(W_s W_t^T) = 4s \frac{1-p}{(K-1)^2} \left(p + \frac{K-3}{2} \right) \begin{pmatrix} (K+2) & 0 & \dots & 0 \\ 0 & 2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 2 \end{pmatrix}$$

Theorem (G-N, 2020)

Let denote $\widehat{S}_n = S_n - \mathbb{E}(S_n)$ and $a = \frac{Kp-1}{K-1}$. If $p > \frac{K+2\theta-1}{2\theta K}$, then we have almost sure convergence

$$\lim_{n\to\infty} \frac{\widehat{S}_n}{n^{a\theta}} = L,$$

where the limiting value L is a non-degenerated random vector. We also have mean square convergence

$$\lim_{n\to\infty} \mathbb{E}\left(\left\|\frac{\widehat{S}_n}{n^{a\theta}} - L\right\|^2\right) = 0.$$

Theorem (G-N, 2020)

The expected value of L is $\mathbb{E}(L) = 0$, while its covariance matrix is obtained by

$$\mathbb{E}(LL^{\top}) = \lim_{n\to\infty} \frac{\Gamma(n)^2}{\Gamma(a\theta + n)^2}\mathbb{E}(\widehat{S}_n\widehat{S}_n^{\top}),$$

Theorem (G-N, 2020)

The expected value of L is $\mathbb{E}(L) = 0$, while its covariance matrix is obtained by

$$\mathbb{E}(LL^T) = \lim_{n\to\infty} \frac{\Gamma(n)^2}{\Gamma(a\theta + n)^2} \mathbb{E}(\widehat{S}_n \widehat{S}_n^T),$$

where

$$\mathbb{E}(\widehat{S}_n \widehat{S}_n^T) = \prod_{i=1}^{n-1} \left(1 + \frac{2a\theta}{i}\right) \mathbb{E}(\widehat{S}_1 \widehat{S}_1^T) + \sum_{i=1}^{n-2} \prod_{k=1}^{n-i} \left(1 + \frac{2a\theta}{n+1-k}\right) \Bigg[\frac{\theta}{d} I_d + (1-\theta) M_p$$
$$- \left(\frac{a\theta}{i} \prod_{l=1}^{i-1} \gamma_{i-l} \mathbb{E}(S_1) + (1-\theta) v_p\right) \left(\frac{a\theta}{i} \prod_{l=1}^{i-1} \gamma_{i-l} \mathbb{E}(S_1) + (1-\theta) v_p\right)^T \Bigg]$$
$$+ \frac{\theta}{d} I_d + (1-\theta) M_p + \prod_{k=1}^{n} \left(1 + \frac{2a\theta}{n+1-k}\right) \mathbb{E}(\widehat{X}_1 \widehat{X}_1^T)$$
$$- \left(\frac{a\theta}{n-1} \prod_{l=1}^{n-2} \gamma_{n-1-l} \mathbb{E}(S_1) + (1-\theta) v_p\right) \left(\frac{a\theta}{n-1} \prod_{l=1}^{n-2} \gamma_{n-1-l} \mathbb{E}(S_1) + (1-\theta) v_p\right)^T.$$

Sketch of the proofs

Let denote $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ the σ-field generated by the sequence $X_1, \ldots, X_n$.

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Therefore, defining $N(n, x) = |\{i \in \{1, \dots, n\} : X_i = x\}|$, the number of steps in the direction $x \in E_d$ until time n, we obtain

$$P(X_{n+1} = x | \mathcal{F}_n) = \begin{cases} p + \theta \left(\frac{1-Kp}{K-1}\right)\left(1 - \frac{N(n,e_1)}{n}\right) & , \text{ if } x = e_1, \\ \frac{1-p}{K-1} + \theta \left(\frac{1-Kp}{K-1}\right)\frac{N(n,x)}{n} & , \text{ if } x \neq e_1. \end{cases}$$

The position of the walker can be obtained by using an auxiliary process, which evolves as an urn model with K colors.

In this sense,

$$S_n = \begin{cases} (U_{1,n} - U_{2,n}, U_{3,n} - U_{4,n}, \dots, U_{K-1,n} - U_{K,n}) & \text{, if } K \text{ is even,} \\ (U_{1,n} - U_{2,n}, U_{3,n} - U_{4,n}, \dots, U_{K-2,n} - U_{K-1,n}) & \text{, if } K \text{ is odd,} \end{cases}$$

where $U_n = (U_{1,n}, \dots, U_{K,n})$ is the vector that denotes the number of balls of each of the K colors, at time n. Each color is associated to the random variables $N(n, x)$ above.

RELATION WITH AN URN MODEL

Then, by defining the random replacement matrix as in Janson (2004), we need to introduce the random vectors ξ_i, for $i \in \{1, \ldots, K\}$, which represent a random number of balls to be added into the urn. Essentially, these column vectors assume values on $\{e_1, \ldots, e_K\}$ the canonical basis of the Euclidean space ^{K}. That is, these vectors denote the color of the ball to be added.

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In this sense, we obtain

$$A = (\mathbb{E}(\xi_1), \ldots, \mathbb{E}(\xi_K)) = \begin{pmatrix} p & p + \theta\frac{1-Kp}{K-1} & \ldots & p + \theta\frac{1-Kp}{K-1} \\ \frac{1-p}{K-1} & \frac{1-p-\theta(1-Kp)}{K-1} & \ldots & \frac{1-p}{K-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1-p}{K-1} & \frac{1-p}{K-1} & \ldots & \frac{1-p-\theta(1-Kp)}{K-1} \end{pmatrix},$$

for this matrix, the largest eigenvalue is $\lambda_1 = 1$, and for $j = 2, \dots, K$ we get

$$\lambda_j = \theta \left(\frac{Kp - 1}{K - 1} \right).$$

Moreover, $u_1 = (1, 1, \cdots, 1)^T$, and

$$v_1 = ((K-1)(p - \lambda_2), 1 - p, \dots, 1 - p)^T \frac{1}{(K-1)(1-\lambda_2)},$$

and, for $j = 2, 3, \dots, K$ we obtain

$$u_j = (1 - p, \cdots, (K-1)\lambda_2 - (K-2) - p, \cdots, 1 - p)^T \frac{1}{(K-1)(1-\lambda_2)},$$

where the different value is at j-th position. Similarly, $v_j = (1, 0, \dots, -1, \dots, 0)^T$, with -1 occupying the j-th position.

We then use Theorem **3.21** from Janson (2004), which states that

$$n^{-1}U_n \longrightarrow \lambda_1 v_1 \ .$$

and Theorem **3.22** of Janson (2004) to prove the functional limit theorem. Then, let $L_I = \{i : \lambda_i < \lambda_1/2\}$ and $L_{II} = \{i : \lambda_i = \lambda_1/2\}$. The limiting covariance matrices are given by

$$\Sigma_I = \sum_{j,k\in L_I} \frac{u_j^T B u_k}{\lambda_1 - \lambda_j - \lambda_k} v_j v_k^T \quad ; \quad \Sigma_{II} = \sum_{j\in L_{II}} u_j^T B u_j v_j v_j^T,$$

where $B = \sum_{i=1}^{K} v_{1i} B_i$ and $B_i = \mathbb{E}[\xi_i \xi_i^T]$,

Therefore,

$$u_i^T B u_j = \frac{1-p}{(K-1)^2(1-\lambda_2)^2} \cdot \begin{cases} p-1 & \text{, if } i \neq j, \\ p-1+(K-1)(1-\lambda_2) & \text{, if } i = j, \end{cases}$$

Therefore,

$$u_i^T B u_j = \frac{1-p}{(K-1)^2(1-\lambda_2)^2} \cdot \begin{cases} p-1 & \text{, if } i \neq j, \\ p-1+(K-1)(1-\lambda_2) & \text{, if } i = j, \end{cases}$$

and

$$v_i v_j^T = \begin{pmatrix} 1 & 0 & \cdots & -1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix},$$

we highlighted j-column and i-row.

We define a locally square-integrable multidimensional martingale, given by

$$M_n = a_n \widehat{S}_n = \sum_{k=1}^{n} a_k \left(\widehat{S}_k - \left(1 + \frac{a\theta}{k-1}\right) \widehat{S}_{k-1} \right) = \sum_{k=1}^{n} a_k \varepsilon_k, \qquad (40)$$

where $\widehat{S}_n = S_n - \mathbb{E}(S_n)$, $a_k = \prod_{l=1}^{k-1} \frac{l}{l + a\theta}$ and $a = \frac{Kp-1}{K-1}$.

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where $\widehat{S}_n = S_n - \mathbb{E}(S_n)$, $a_k = \prod_{l=1}^{k-1}\frac{l}{l+a\theta}$ and $a = \frac{Kp-1}{K-1}$.

Then, as in Theorem 3.7 in Bercu (2018), we need to prove that

$$\lim_{n\to\infty} \mathrm{Tr}\langle M\rangle_n < \infty \quad \text{a.s} \qquad (41)$$

where TrA stands for the trace of matrix A and

$$\langle M\rangle_n = \sum_{k=1}^{n}\mathbb{E}\left[(a_k\varepsilon_k)(a_k\varepsilon_k)^T\middle|\mathcal{F}_{k-1}\right]. \qquad (42)$$

Then,

$$\begin{aligned}\mathrm{Tr}\langle M\rangle_n &= a_1^2\mathbb{E}(\varepsilon_1\varepsilon_1^T) + \alpha(p,\theta,K)\sum_{l=1}^{n-1} a_{l+1}^2\left(1 - \frac{2\theta(1-\theta)}{\alpha(p,\theta,K)}\frac{\mathrm{Tr}(S_l v_p^T)}{l}\right) \\ &\quad - a^2\theta^2\sum_{l=1}^{n-1}\left(\frac{a_{l+1}}{l}\right)^2\|S_l\|^2,\end{aligned} \tag{43}$$

where $\alpha(p,\theta,K) = 1 - (1-\theta)^2\left(p\frac{(1-p)^2}{K-1}\right)$.

Then,

$$\begin{aligned} \mathrm{Tr}\langle M\rangle_n &= a_1^2\mathbb{E}(\varepsilon_1\varepsilon_1^T) + \alpha(p,\theta,K)\sum_{l=1}^{n-1} a_{l+1}^2\left(1 - \frac{2\theta(1-\theta)}{\alpha(p,\theta,K)}\frac{\mathrm{Tr}(S_l v_p^T)}{l}\right) \\ &\quad - a^2\theta^2\sum_{l=1}^{n-1}\left(\frac{a_{l+1}}{l}\right)^2 \|S_l\|^2, \end{aligned} \tag{43}$$

where $\alpha(p,\theta,K) = 1-(1-\theta)^2\left(p\frac{(1-p)^2}{K-1}\right)$.

In addition, note that for all $l \geq 1$ and for all p, K and θ, $-1 \leq \frac{2\theta(1-\theta)}{\alpha(p,\theta,K)}\frac{\mathrm{Tr}(S_l v_p^T)}{l} \leq 1$. Then,

$$\mathrm{Tr}\langle M\rangle_n \leq a_1^2\mathbb{E}(\varepsilon_1\varepsilon_1^T) + 2\alpha(p,\theta,K)\sum_{l=1}^{n-1} a_{l+1}^2. \tag{44}$$

Note that, $\sum_{l=1}^{n} a_l^2 = \sum_{l=1}^{n} \left(\frac{\Gamma(a\theta + 1)\Gamma(l)}{\Gamma(a\theta + l)} \right)^2$, which in the superdiffusive regime satisfies

$$\lim_{n\to\infty} \sum_{l=1}^{n} a_l^2 = \sum_{l=1}^{\infty} \left(\frac{\Gamma(a\theta + 1)\Gamma(l)}{\Gamma(a\theta + l)} \right)^2 = {}_3F_2\left(1, 1, 1; a\theta + 1, a\theta + 1; 1\right), \tag{45}$$

the finite confluent hypergeometric function. Therefore,

$$\lim_{n\to\infty} \mathrm{Tr}\langle M \rangle_n < \infty \quad \text{a.s}\ . \tag{46}$$

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Muchas gracias!