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GALOIS CLOSURE OF A FIVEFOLD COVERING AND DECOMPOSITION OF ITS JACOBIAN

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Abstract. For a covering map of degree 5 between compact Riemann surfaces, every possible Galois closure is determined in terms of the ramification data of the map. Since the group that acts on the Galois closure also acts on the Jacobian variety of the covering surface, we describe its group algebra decomposition in terms of the Jacobian and Prym varieties of the intermediate coverings of the Galois closure.

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Introduction

In the moduli space $\mathcal{A}_g$ of isomorphism classes of principally polarized abelian varieties of dimension g, the subspace of isomorphism classes of Jacobian varieties deserve special attention. Due to Torelli's theorem, to each curve X of genus g we can biunivocally associate a principally polarized abelian variety $\mathrm{Jac}(X)$ of dimension g; in this manner, we can think the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g as a special subspace of $\mathcal{A}_g$, the subspace of the Jacobian varieties. This particular kind of principally polarized abelian varieties were the first ones to be studied and they are still the best known among the latter ones due to the information we can naturally get from their associated curve.

Smooth projective algebraic curves over $\mathbb{C}$ (or, equivalently, compact Riemann surfaces) and rational (equivalently, holomorphic) maps between them are one of the first, simplest and most interesting objects of study in algebraic geometry. This is not only for their self charm, but from the direct implications their properties have in other areas of geometry, in particular, and mathematics, in general. As said before, abelian varieties first arose in the study of curves.

The decomposition of abelian varieties into simple abelian subvarieties is archived by the Poincare's complete reducibility theorem (see [**2**]), and it is unique up to isomorphism; however, there is no easy manner to make that decomposition explicit. Decomposable abelian varieties are an old and studied topic (see [**27**]). When we have a group action on an abelian variety, we have its isotypical decomposition, which is usually coarser than the former one, into G-invariant abelian sub-varieties and the, slightly finer, group algebra decomposition. Moreover, in a Jacobian variety $\mathrm{Jac}(X)$, when this action is inherited by an action on X, we can express the pieces of the group algebra decomposition as Prym varieties of intermediate coverings of the Galois covering induced by the group action on X. The first and founder example of this phenomena is the Recillas trigonal construction (see [**19**]) that shows that the Jacobian of a tetragonal curve is isomorphic to the Prym variety of a double cover of a trigonal curve. Later, in [**20**] and [**21**], Recillas and Rodriguez generalized these results for curve coverings of degree 3 and 4, respectively. They also analyze the polarization types of the isogenies involved. In this article we study analogous results for the case of degree 5 coverings in the light of the more general results of [**7**]. It also should be mentioned that actions of $\mathfrak{A}_5$ and $\mathfrak{S}_5$ on Jacobian varieties are also studied on [**15**, **24**] and actions of dihedral groups on [**6**].

Throughout this article, let $f\colon X \to Y$ denote a ramified covering of degree 5 between compact Riemann surfaces with branch locus B and monodromy representation $\rho\colon\ \pi_1(Y -$

$B) \to \mathfrak{S}_5$. Although f is not necessarily Galois (that is, the quotient of X by a group action), its Galois closure $\hat{f}\colon \hat{X} \to Y$ is. This action induces another one on the Jacobian variety $\mathrm{Jac}(\hat{X})$; hence, we can decompose $\mathrm{Jac}(\hat{X})$ into smaller abelian varieties through the group algebra decomposition (see [**7**]). Since the automorphism group $\mathrm{Aut}(\hat{f})$ is naturally isomorphic to the monodromy group $\mathrm{Mon}(f)$, the geometric signature (see [**23**]) of $\hat{f}$ is determined by the ramification data of f; hence, the group algebra decomposition of $\mathrm{Jac}(\hat{X})$ depends on the ramification data of f.

There are two main results in this article: in Theorems IV.10 and IV.12, we enumerate each possible monodromy group $\mathrm{Mon}(f)$, up to conjugacy in $\mathfrak{S}_5$, in terms of the ramification data of f; we also give explicit generating vectors for each possible case. Then, and as a application, in Theorems V.1, V.2, V.4, V.6 and V.8, we give the group algebra decomposition of $\mathrm{Jac}(\hat{X})$ for each possible $\mathrm{Mon}(f)$. Additionally, in Theorems V.3, V.5 and V.7 we study some special cases where we can explicitly compute the polarization induced by $\mathrm{Jac}(\hat{X})$ in each piece of its group algebra decomposition.

CHAPTER I

Monodromy Representation and Galois Closure of a Covering

1. Covering maps and their automorphisms

Let (M, m) be a *pointed topological surface* and let $F_0\colon (U_0, u_0) \to (M, m)$ be its, unique up to isomorphism, *universal covering* (see [**18**, sections 80 and 82]). The *fundamental group* $\pi_1(M, m)$ (see [**18**, section 52]) acts on U_0 as follows: Choose a loop γ on M based at m and a point $v \in U_0$. Let α be a path on U_0 starting at v and ending at u_0, then $(F_0)_*\alpha$ is a path on M starting at $F_0(v)$ and ending at m. Let $\tilde{\gamma}$ denote the unique lift of the loop γ that starts at u_0, and then lift the reverse path $\overline{(F_0)_*\alpha}$ starting at $\tilde{\gamma}(1)$ to a path β. The endpoint of β, which lies over $F_0(v)$, depends only on v and the homotopy class $[\gamma]$; see [**16**, section V.7] and [**17**, section III.4] for further details. Therefore, $[\gamma] \cdot v = \beta(1)$ is well defined and yields an action of $\pi_1(M, m)$ on U_0 (see Figure 1). Since this action preserves every fiber of F_0, it is not just an action on the space U_0 but on the covering map F_0 (see [**16**, section V.6]).

The above action may be restricted to any subgroup H of the fundamental group $\pi_1(M, m)$, and from this restricted action arises a *quotient space*, namely U_0/H, and a covering map $\pi_H\colon (U_0, u_0) \to (U_0/H, [u_0])$, which we call *quotient map* associated to H, given by the natural projection (see [**16**, Lemma 10.1 on p. 173]). Since F_0 respect the fibers of π_H, it naturally induces a covering map $\pi^H\colon (U_0/H, [u_0]) \to (M, m)$, which we call *induced map* associated to H, such that the following diagram commutes:

$$\begin{array}{ccc} (U_0, u_0) & \xrightarrow{\ \pi_H\ } & \left(\dfrac{U_0}{H}, [u_0]\right) \\ {\scriptstyle F_0}\big\downarrow & \swarrow{\scriptstyle \pi^H} & \\ (M, m) & & \end{array} \qquad \text{(I.1)}$$

Conversely, the induced homomorphism $F_*\colon \pi_1(U, u) \to \pi_1(M, m)$ (see [**18**, p. 333]) is injective for any covering map $F\colon (U, u) \to (M, m)$, and two of them are related by the following result.

Proposition I.1 ([**16**, Theorem 6.6 on p. 159])**.** *Two covering spaces $F_1\colon U_1 \to M$ and $F_2\colon U_2 \to M$ are isomorphic if and only if, for any two points $u_1 \in U_1$ and $u_2 \in U_2$ such*

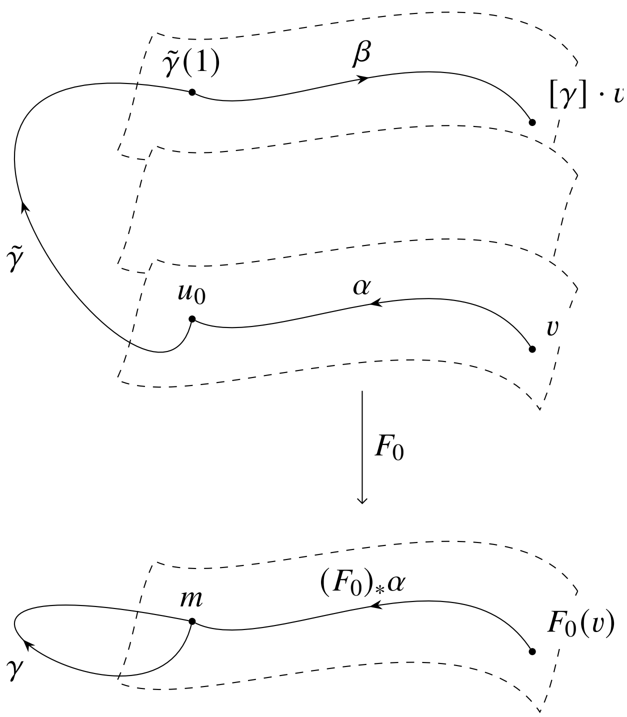


FIGURE 1. Action of the fundamental group $\pi_1(M,m)$ on the universal covering $F_0\colon (U_0,u_0)\to(M,m)$

that $F_1(u_1)=F_2(u_2)=m$, the subgroups $(F_1)_*\,\pi_1(U_1,u_1)$ and $(F_2)_*\,\pi_1(U_2,u_2)$ belong to the same conjugacy class in $\pi_1(M,m)$.

In particular, Theorem I.1 implies that the induced covering map $\pi^{F_*\,\pi_1(U,u)}$ is isomorphic to $F\colon (U,u)\to(M,m)$. Summarizing, and according to [**16**, Lemma 6.3 on p. 159], we get the following result.

Proposition I.2. *There is a one-to-one correspondence between*:

(1) *isomorphism classes of connected coverings $F\colon (U,u)\to(M,m)$*; *and*
(2) *conjugacy classes of subgroups H of $\pi_1(M,m)$.*

Moreover, the degree of such a covering F is exactly the index in $\pi_1(M,m)$ of the corresponding subgroup H whenever one of them is finite.

In the rest of this section, we deal with Galois coverings; that is, covering maps whose automorphism group acts transitively on its fibers. Let H be a subgroup of $\pi_1(M,m)$ associated by Theorem I.2 to a covering map $F\colon U\to M$. Consider the normalizer $\mathrm{N}_{\pi_1(M,m)}(H)$ of H in $\pi_1(M,m)$; in the same manner that $\pi_1(M,m)$ acts on the universal covering U_0 of M, an automorphism of F arises from each homotopy class $[\gamma]\in\mathrm{N}_{\pi_1(M,m)}(H)$. This association is not injective but, according to [**16**, Corollary 7.3 on p. 163], yields an isomorphism

$$\mathrm{Aut}(F) \cong \frac{\mathrm{N}_{\pi_1(M,m)}(H)}{H},$$

where $\mathrm{Aut}(F)$ denotes the automorphism group of the covering map $F\colon U \longrightarrow M$ (see [**16**, pp. 158–159]).

In the special case where H is normal in $\pi_1(M,m)$, the associated covering is called *Galois* (or *regular*) and the latter equation simplifies to

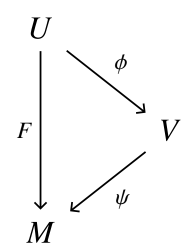
$$\mathrm{Aut}(F) \cong \frac{\pi_1(M,m)}{H}.$$

Moreover, as stated in [**16**, Lemma 8.1 on p. 164], the covering $F\colon U \longrightarrow M$ is Galois if and only if $\mathrm{Aut}(F)$ operates transitively on any of its fibers; the latter condition may be rephrased as $U/\mathrm{Aut}(F) \cong M$.

Conversely, if a group G acts properly discontinuously on a surface Z, then the natural projection $\pi_G\colon Z \longrightarrow Z/G$ is a Galois covering and $\mathrm{Aut}(\pi_G) = G$ (see [**16**, Proposition 8.2 on p. 165]). Thereby, a covering map is Galois if and only if it is a quotient by a properly discontinuous group action.

Definition I.1. An *intermediate cover* of a covering map $F\colon U \longrightarrow M$ is a space V for which exists two covering maps $\phi\colon U \longrightarrow V$ and $\psi\colon V \longrightarrow M$ such that $F = \psi \circ \phi$. The maps ϕ and ψ are called *intermediate coverings*.

Proposition I.3. *Let $F\colon U \longrightarrow M$ be a Galois covering. Then, for every intermediate cover as in the following diagram*:



the map ϕ is also Galois.

Proof. The subgroup $H = F_* \pi_1(U,u) = \psi_* \phi_* \pi_1(U,u)$, associated to F by Theorem I.2, is normal in $\pi_1(M,m)$ and then, according to [**22**, 1.4.6 on p. 20], the group $\phi_* \pi_1(U,u)$ is normal in $\psi_*^{-1} \pi_1(M,m)$, which is precisely $\pi_1(V,v)$. □

2. Galois closure of a covering map

Let (M,m) be a pointed surface and let $F\colon (U,u) \longrightarrow (M,m)$ be a connected covering of finite degree d with associated subgroup H, then $[\pi_1(M,m) : H] = d$.

Following [**17**, section III.4], we define a homomorphism $\rho\colon \pi_1(M,m) \longrightarrow \mathfrak{S}_d$, called *monodromy representation*, as follows: Fix an enumeration $u_1, \ldots, u_d$ for the fiber $F^{-1}(m)$;

then, for every loop γ in M based at m and for each $i \in \{1, \dots, d\}$, there is a lifted path $\tilde{\gamma}_i$ starting at u_i. Since each endpoint $\tilde{\gamma}_i(1)$ lies on the fiber $F^{-1}(m)$, there is an index $\sigma_\gamma(i) \in \{1, \dots, d\}$ such that $\tilde{\gamma}_i(1) = u_{\sigma_\gamma(i)}$; in this manner, a function $\sigma_\gamma \colon \{1, \dots, d\} \to \{1, \dots, d\}$ is defined. Repeating this procedure with the reverse path $\overline{\gamma}$ yields the inverse of the map σ_γ, thus σ_γ is a permutation of $\{1, \dots, d\}$ and, according to [**16**, Lemma 3.3 on p. 152], this permutation depends only on the homotopy class $[\gamma]$. Besides, this procedure clearly respects the concatenation of paths; that is, $\sigma_{\gamma * \delta} = \sigma_\gamma \sigma_\delta$. Therefore, the assignment $\rho([\gamma]) = \sigma_\gamma$ defines the desired homomorphism. Although ρ depends on the chosen enumeration of $F^{-1}(m)$, a different choice gives a conjugate representation; therefore ρ is unique up to conjugation in $\mathfrak{S}_d$. Its image is called the *monodromy group* of F, which we will denote by $\mathrm{Mon}(F)$. The connectedness of U implies that $\mathrm{Mon}(F)$ is a transitive subgroup of $\mathfrak{S}_d$, the symmetric group of degree d (see [**17**, Lemma 4.4 on p. 87]).

Conversely, given a homomorphism $\rho\colon \pi_1(M, m) \to \mathfrak{S}_d$ with transitive image G, consider the subgroup $\mathrm{Stab}_G(1)$ of G. According to [**22**, 1.6.1.i on p. 31], we have that $[G : \mathrm{Stab}_G(1)] = \#\{1, \dots, d\} = d$; thus, if H denotes $\rho^{-1}\,\mathrm{Stab}_G(1)$, then

$$[\pi_1(M, m) : H] = \frac{[\pi_1(M, m) : \ker \rho]}{[H : \ker \rho]} = [G : \mathrm{Stab}_G(1)] = d.$$

Therefore, in the notation of section 1, the map $F_H \colon U_0/H \to M$ is a covering of degree d. Moreover, a direct computation shows that the $\mathrm{Mon}(F_H)$ is conjugate to G itself (see [**17**, p. 89]).

Summarizing, we get the following result.

Proposition I.4. *There is a one-to-one correspondence between*:

(1) *isomorphism classes of connected coverings* $F\colon (U, u) \to (M, m)$ *of degree* d; *and*
(2) *group homomorphisms* $\rho\colon\ \pi_1(M, m) \to \mathfrak{S}_d$ *with transitive image up to conjugacy in* $\mathfrak{S}_d$.

Definition I.2. The *Galois closure* of a covering map $F\colon U \to M$ with monodromy representation $\rho\colon\ \pi_1(M, m) \to \mathfrak{S}_d$ is the map associated to $\ker \rho$ by Theorem I.2, and is denoted by $\hat{F}\colon \hat{U} \to M$.

Since $\ker \rho$ is a normal subgroup of $\pi_1(M, m)$, the covering map $\hat{F}$ is Galois; this fact justifies Definition I.2. Moreover, according to Theorem I.3, every intermediate covering of $\hat{F}$ is also Galois. Since $\pi_1(M, m)/\ker \rho \cong \mathrm{Mon}(F)$, we get the following result, which relates $\mathrm{Mon}(F)$ with $\hat{F}$.

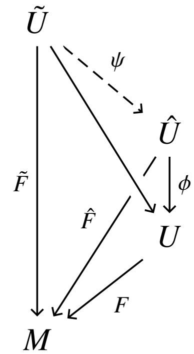
Proposition I.5. *For every covering map* $F\colon U \to M$*, we have* $\mathrm{Aut}(\hat{F}) \cong \mathrm{Mon}(F)$.

On the minimality of the Galois closure, we know the following result.

Theorem I.6. *Given a covering map* $F\colon U \to M$*, its Galois closure* $\hat{F}\colon \hat{U} \to M$ *is the minimal Galois covering that factors through* F*; that is, every Galois covering that factors through* F *also factors through* $\hat{F}$.

Proof. If $\rho\colon \pi_1(M,m) \to \mathfrak{S}_d$ denotes the monodromy representation of F, then $\ker\rho = \operatorname{Core}_{\pi_1(M,m)}(F_* \pi_1(U,u))$, see [**22**, p. 16]. Indeed, we have $[\gamma] \in \ker\rho$ if and only if every lift of γ is a loop in U; that is, we have $[\gamma] \in F_* \pi_1(U,\upsilon)$ for each $\upsilon \in F^{-1}(m)$. But, according to [**16**, Theorem 4.2 on p. 155], the family $\{ F_* \pi_1(U,\upsilon) : \upsilon \in F^{-1}(m) \}$ is a whole conjugacy class in $\pi_1(M,m)$; more precisely, the conjugacy class of $F_* \pi_1(U,u)$. That is, in turn, equivalent to $[\gamma] \in \operatorname{Core}_{\pi_1(M,m)} F_* \pi_1(U,u)$.

Let $\tilde{F}\colon (\tilde{U},\tilde{u}) \to (M,m)$ be a Galois covering which factors through F. Recall that the correspondence in Theorem I.2 is order-reversing, so $\tilde{F}_* \pi_1(\tilde{U},\tilde{u}) \subseteq F_* \pi_1(U,u)$. Since $\tilde{F}_* \pi_1(\tilde{U},\tilde{u})$ is normal in $\pi_1(M,m)$, we have $\tilde{F}_* \pi_1(\tilde{U},\tilde{u}) \subseteq \operatorname{Core}_{\pi_1(M,m)}(F_* \pi_1(U,u)) = \ker\rho$. Therefore, Theorem I.2 yields the existence of a covering map ψ such that the following diagram commutes:



□

Before ending this section, it should be noted that the Galois closure $\hat{F}$ of a degree d covering map F is also of finite degree; indeed, we have that $\deg \hat{F} = [\pi_1(M,m) : \ker\rho] = |\mathrm{Mon}(F)| \leq d!$.

Now we carry the theory of topological coverings to the context of *holomorphic maps* between *compact Riemann surfaces*. Let $f\colon X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces (see [**17**, section II.3]). The set B of branch values of f, namely the *branch locus* of f, is a discrete subset of Y; thus, it is a finite set. The restriction of f to $X - f^{-1}(B)$ is a covering map in the topological sense, and it will be denoted by $\tilde{f}$ while its domain and image by $\tilde{X}$ and $\tilde{Y}$, respectively. The original map f is often called a *ramified covering map* (see [**1**, subsection I.3.20]) (or just covering map, when there is no room for confusion). The degree of $\tilde{f}$ as a covering map clearly coincides with the degree of f as a holomorphic map (see [**17**, Proposition 4.8 on p. 47]) and it will be denoted by d.

Choose a base point $y \in \tilde{Y}$. The monodromy representation $\rho\colon \pi_1(\tilde{Y},y) \to \mathfrak{S}_d$ of the covering $\tilde{f}$ will be also called the *monodromy representation* of f and $\operatorname{im}\rho$ its *monodromy group*, denoted also by $\mathrm{Mon}(f)$. Since $\tilde{Y}$ is connected, we have that $\mathrm{Mon}(f)$ is a transitive subgroup of $\mathfrak{S}_d$. Although $\mathrm{Mon}(f)$ depends on the base point $y \in \tilde{Y}$, a different choice yields a conjugate monodromy group.

Following the construction in [**17**, p. 88], a special kind of loop will be defined for each branch value $b \in B$. Let W be an open neighborhood of b such that $f^{-1}(W - b)$ decomposes

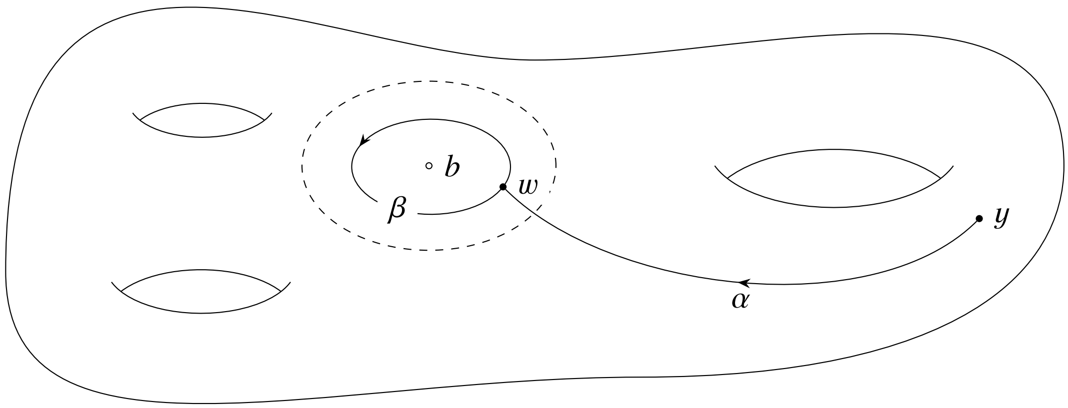


FIGURE 2. Small loop around b on the Riemann surface $\tilde{Y}$

as a disjoint union of punctured discs. Fix a point $w \in W - \{b\}$, then choose a path α from y to w and a loop β in $W - \{b\}$ based at w with winding number 1 around b, see Figure 2.

DEFINITION I.3. In the previous notation, the loop $\alpha^{-1} * \beta * \alpha$ on $\tilde{Y}$ based at y is called a *small loop* around b and it will be denoted by γ_b.

Note that neither γ_b nor its homotopy class $[\gamma_b]$ are determined just by b but also by the choice of α; however, different paths α yields $\pi_1(\tilde{Y}, y)$-conjugate homotopy classes. The following result concerning small loops will be very useful.

Theorem I.7 ([**17**, Lemma 4.6 on p. 88])**.** *Suppose that above the branch value $b \in Y$ there are k preimages $x_1, \ldots, x_k$, and that* $\operatorname{mult}_{x_j}(f) = \nu_j$. *The cycle structure of the permutation $\rho(\gamma_b)$ representing the class of a small loop around b is $[\nu_1, \ldots, \nu_k]$.*

In the same way as in section 2, coverings onto a given Riemann surface may be classified in terms of their monodromy representation. For this purpose, fix a compact Riemann surface Y. According to [**17**, Lemma 4.7 on p. 89], for any topological covering $\phi\colon W \to Y$ there is a unique complex structure on W such that ϕ is a holomorphic map. Thereby, Theorem I.4 directly extend to ramified coverings.

Furthermore, given a finite subset B of Y, Theorem I.2 remains true for the Riemann surface $Y - B$, and, according to [**10**, Lemma 1.80 on p. 64], each map $\tilde{f}\colon \tilde{W} \to Y - B$ of this correspondence can be holomorphically extended in an (up to isomorphism) unique manner to a map $f\colon X \to Y$, where X is a compact Riemann surface. These facts summarize in the following theorem.

Theorem I.8 ([**17**, Proposition 4.9 on p. 91])**.** *Let Y be a compact Riemann surface, let B be a finite subset of Y, and let y be a base point of $Y - B$. Then there is a one-to-one correspondence between*:

(1) *isomorphism classes of holomorphic maps* $f\colon X \to Y$ *of degree* d *whose branch points lie in* B; *and*
(2) *group homomorphisms* $\rho\colon \pi_1(Y - B, y) \to \mathfrak{S}_d$ *with transitive image up to conjugacy in* $\mathfrak{S}_d$.

For each point $b \in B$, *let* γ_b *be the class of a small loop on* $Y - B$ *around* b *based at* y. *If* $\rho(\gamma_b)$ *has cycle structure* $[\nu_1, \dots, \nu_k]$, *then there are* k *preimages* $x_1, \dots, x_k$ *of* b *in the corresponding cover* $f\colon X \to Y$ *with* $\operatorname{mult}_{x_j}(f) = \nu_j$ *for each* $j \in \{1, \dots, k\}$.

Given a holomorphic map $f\colon X \to Y$ of degree d and a branch value b in Y, there is an enumeration $x_1, \dots, x_k$ of the fiber $f^{-1}(b)$ such that $\operatorname{mult}_{x_1}(f) \geq \cdots \geq \operatorname{mult}_{x_k}(f)$. This enumeration is not unique but the tuple $[\operatorname{mult}_{x_1}(f), \dots, \operatorname{mult}_{x_k}(f)]$ is.

Definition I.4. With the notation of Theorem I.8, the *type* of a branch value b is the tuple of integers $[\operatorname{mult}_{x_1}(f), \dots, \operatorname{mult}_{x_k}(f)]$.

Note that, although the class γ_b of a small loop around b is defined only up to conjugacy, the cycle structure of $\rho(\gamma_b)$ is well defined; moreover, according to Theorem I.7, it coincides with the type of the branch value b. The branch value b will be called *odd* or *even* according to the parity of $\rho(\gamma_b)$.

Definition I.5. The *ramification data* of f is the set of branch values of f and their respective types. We also call ramification data, when there is no place for ambiguity, to the tuple of ramification types of the branch values of f.

Let g_Z denote the *genus* of a compact Riemann surface Z. Note that the ramification data is just a way of organizing the information given by the *ramification divisor* of f, denoted by R_f (see [**17**, Definition 1.18 on p. 134]); the genus of X can be computed from the ramification data of f and the genus of Y through Riemann–Hurwitz formula, stated in the following proposition.

Proposition I.9 ([**17**, Theorem 4.16 on p. 52])**.** *Let* $f\colon X \to Y$ *be a nonconstant holomorphic map between compact Riemann surfaces. Then*

$$2g_X - 2 = \deg(f)(g_Y - 2) + \sum_{p \in X} [\operatorname{mult}_p(f) - 1].$$

We call $\sum_{p\in X} [\operatorname{mult}_p(f) - 1]$ the *total ramification* of f and denote it by $\deg(R_f)$, see also [**17**, Definition 1.2 on p. 129].

As for topological coverings, the map $f\colon X \to Y$ is called *Galois* or *regular* if its associated subgroup H is normal in $\pi_1(\tilde{Y}, y)$. According to [**10**, Lemma 1.80 and Proposition 1.81 on p. 64], each automorphism of $\tilde{f}$ can be uniquely extended to an automorphism of f; thus, we have the following isomorphism:

$$\operatorname{Aut}(f) \cong \frac{\mathrm{N}_{\pi_1(\tilde{Y}, y)}(H)}{H}. \tag{I.2}$$

Thereby, $\operatorname{Aut}(f) \cong \pi_1(\tilde{Y}, y)/H$ if and only if f is Galois. A holomorphic map is Galois if and only if it is induced by a properly discontinuous holomorphic group action, see [**17**, Definition 3.10 on p. 83].

Following Definition I.2, denote the Galois closure of $\tilde{f}$ by $\varphi\colon W \to Y - B$. As before, W can be considered as a Riemann surface and φ as a holomorphic map in a unique manner; moreover, according to [**10**, Lemma 1.80 on p. 84], the map φ can be uniquely extended to a holomorphic map $\hat{f}\colon \hat{X} \to Y$ between compact Riemann surfaces. This map $\hat{f}$ is called *Galois closure* of f and it is, indeed, Galois. Also the previous remarks on the automorphisms of a holomorphic implies that Theorem I.5 and Theorem I.6 are also valid for compact Riemann surfaces and (ramified) coverings.

Note that, by construction, every ramification point of $\hat{f}$ is a preimage of a branch value of the original map f. Moreover, according to [**17**, pp. 75–76], the stabilizers of the preimages of a fixed branch value $b \in B$ are a full conjugate class of cyclic subgroups of $\operatorname{Aut}(\hat{f})$ of order $\operatorname{mult}_p(\hat{f})$ where $p \in \hat{f}^{-1}(b)$.

3. Geometric signature of a Galois covering

Throughout this section, let G be a finite group that acts *holomorphically* and *effectively* on a compact Riemann surface Z (see [**17**, section III.3]). The *stabilizer* and *orbit* of a point $p \in Z$ are denoted by $\operatorname{Stab}_G(p)$ and $G \cdot p$, respectively. According to [**17**, Propositions 3.1 and 3.2], the points with non trivial stabilizers are discrete (so finite) and each stabilizer is a cyclic subgroup of G. Moreover, the quotient map $\pi_G\colon Z \to Z/G$ has degree $|G|$ and $\operatorname{mult}_p(\pi_G) = |\operatorname{Stab}_G(p)|$ for each $p \in Z$ (see [**17**, Theorem 3.4 on p. 78]).

When G is the automorphism group of the Galois closure of a covering $f\colon X \to Y$, Theorem I.5 states that $G \cong \operatorname{Mon}(f)$; in that notation, we have the following characterization of the stabilizer of a point $p \in Z$, which will be very useful.

Theorem I.10. *Consider a covering map $f\colon X \to Y$ with Galois closure $\hat{f}\colon \hat{X} \to Y$. For each ramification point $p \in \hat{X}$ of $\hat{f}$, there is a small loop $\gamma_{\hat{f}(p)}$ around $\hat{f}(p)$ such that the group isomorphism of Theorem I.5 restricts to*

$$\operatorname{Stab}_{\operatorname{Aut}(\hat{f})}(p) \cong \langle \rho([\gamma_{\hat{f}(p)}]) \rangle,$$

where ρ is the monodromy representation of f.

Proof. According to [**17**, Corollary 3.5 on p. 79], there is a generator g in $\operatorname{Stab}_{\operatorname{Aut}(\hat{f})}(p)$ and a local coordinate z on $\hat{X}$ centered at p such that $g(z) = \mathrm{e}^{2\pi\mathrm{i}/m} z$ where $m = \operatorname{mult}_p(\hat{f})$. Assume that the range of z contains the closed disc D of radius 1 centered at 0 and that its domain U is sufficiently small such that $U \cap gU = \varnothing$ for each $g \in \operatorname{Aut}(\hat{f}) - \operatorname{Stab}_{\operatorname{Aut}(\hat{f})}(p)$ and that no point of the (topological) punctured disc $U - \{p\}$ is fixed by any element of $\operatorname{Stab}_{\operatorname{Aut}(\hat{f})}(p)$ (see [**17**, Proposition 3.3 on p. 77]).

Set a path $\alpha\colon [0,1] \to \hat{X}$ such that $z(\alpha(t)) = \mathrm{e}^{2\pi\mathrm{i}t/m}$; thus $\operatorname{range}(\alpha) \subset U$. As in the proof of [**17**, Corollary 3.5], we can choose a local coordinate w near $\hat{f}(p)$ such that the formula of $\hat{f}$ is $w = z^m$ it these coordinates. Set $\gamma_{\hat{f}(p)} = \hat{f}_*\alpha$, then $\gamma_{\hat{f}(p)}$ is a loop in Y and $w(\gamma(t)) = \mathrm{e}^{2\pi \mathrm{i} t}$; so it is a small loop around $\hat{f}(p)$. If h is the automorphism of $\hat{f}$ associated to $\rho([\gamma_{\hat{f}(p)}])$ by Theorem I.5, then, in the local coordinate z, the point $h(1)$ is the end of the unique lift of $\gamma_{\hat{f}(p)}$ starting at $z = 1$, namely α; so $h(1) = z(\alpha(1)) = \mathrm{e}^{2\pi\mathrm{i}/m}$. Since $h(1) \in D$, we have that $h \in \mathrm{Stab}_{\mathrm{Aut}(\hat{f})}(p)$. Since $h(1) = g(1)$ and no point of U except p is fixed by any element of $\mathrm{Stab}_{\mathrm{Aut}(\hat{f})}(p)$, we conclude that $g = h$. But $\langle g\rangle = \mathrm{Stab}_{\mathrm{Aut}(\hat{f})}(p)$, so we are done. □

Since the number of points in the orbit of a point $p \in Z$ is given by $|G \cdot p| = |G|/|\mathrm{Stab}_G(p)|$ and if $q = \sigma(p)$ for some $\sigma \in G$, then $\mathrm{Stab}_G(q) = \mathrm{Stab}_G(p)^\sigma$; the branch value $\pi_G(p)$ is of type

$$\big[\underbrace{|\mathrm{Stab}_G(p)|, \dots, |\mathrm{Stab}_G(p)|}_{[G\colon\ \mathrm{Stab}_G(p)]\text{ times}}\big].$$

Therefore, if $\{p_1, \dots, p_k\}$ is a set with exactly one point in Z from each orbit with nontrivial stabilizer, then the total ramification $\deg(R_{\pi_G})$ can be computed from the tuple $\big(|\mathrm{Stab}_G(p)|\big)_{p=1}^k$; indeed,

$$\begin{aligned}\deg(R_{\pi_G}) &= \sum_{i=1}^k \frac{|G|}{|\mathrm{Stab}_G(p_i)|}\big(|\mathrm{Stab}_G(p_i)| - 1\big)\\ &= |G| \sum_{i=1}^k \left(1 - \frac{1}{|\mathrm{Stab}_G(p_i)|}\right).\end{aligned}$$

With this notation, the genus g_Z can be computed in terms of the tuple $\big(|\mathrm{Stab}_G(p)|\big)_{p=1}^k$ and $g_{Z/G}$ as in the following result.

Theorem I.11 ([**17**, Corollary 3.7 on p. 80])**.** *Let G be a finite group acting holomorphically and effectively on a compact Riemann surface Z, with quotient map $\pi_G\colon Z \to Z/G$. Suppose $\{p_1, \dots, p_k\}$ is a set with exactly one point in Z from each G-orbit with nontrivial stabilizer. Then*

$$2g_Z - 2 = |G| \left[2g_{Z/G} - 2 + \sum_{i=1}^k \left(1 - \frac{1}{|\mathrm{Stab}_G(p_i)|}\right)\right].$$

This motivate the following definition.

Definition I.6. For a branched covering $\pi_G\colon Z \to Z/G$ and a subset $\{p_1, \dots, p_k\}$ of Z with a single point from each orbit with nontrivial stabilizer the vector of numbers

$$\big(g_{Z/G}; |\mathrm{Stab}_G(p_1)|, \dots, |\mathrm{Stab}_G(p_k)|\big)$$

is called the *signature* of G on Z.

As points in the same orbit has conjugate stabilizers, the signature of G does not depend on the choice of $\{p_1, \dots, p_k\}$ up to a re-enumeration of the orbits. If we rearrange the orbits in such a way that $|\mathrm{Stab}_G(p_i)| \leq |\mathrm{Stab}_G(p_j)|$ for $i < j$, then the signature of π_G is uniquely determined; thereby, in order to get a well-defined signature, we will impose this additional condition. The following definition, taken from [**23**, Definition 3.1.1], refines Definition I.6.

Definition I.7. Consider a branched covering $\pi_G\colon Z \to Z/G$ and a subset $\{p_1, \dots, p_k\}$ of Z with a single point from each G-orbit with nontrivial stabilizer. Let C_i denote the conjugacy class of $\mathrm{Stab}_G(p_i)$ into G. Then the vector

$$(g_{Z/G}; C_1, \dots, C_k)$$

is called the *geometric signature* of G on Z.

Notation I.1. The conujgacy class of a subgroup H of a group G will be denoted by $\mathrm{Class}_G(H)$, or just by $\mathrm{Class}(H)$ if there is no place to confusion about G.

As points in the same orbit has conjugate stabilizers, the geometric signature is also independent of the chosen $\{p_1, \dots, p_k\}$ up to a re-enumeration of the orbits; but unlike the previous case, there is no standard manner to order two different classes of stabilizers with the same order, so it is more suitable to think the geometric signature as a *multiset* than a tuple.

The following definition and theorem state precisely when a group G acts on a Riemann surface of a given genus. The theorem is also known as the *Riemann's Existence Theorem*.

Definition I.8 ([**4**, Definition 2.2]). A $(2g+r)$ tuple $(a_1, \dots, a_g, b_1, \dots, b_g, c_1, \dots, c_r)$ of elements of a group G is called a $(g; m_1, \dots, m_r)$-vector if

$$\prod_{i=1}^{g} [a_i, b_i] \prod_{i=1}^{r} c_i = 1$$

and $|c_i| = m_i$ for each $i = 1, \dots, r$. The vector is called *generating* $(g; m_1, \dots, m_r)$*-vector* if G is generated by $\{\, a_i, b_j, c_k : i, j \in \{1, \dots, n\}, k \in \{1, \dots, r\} \,\}$.

Theorem I.12 ([**4**, Proposition 2.1])**.** *The group G acts on a Riemann surface Z of genus g_Z with signature $(g; m_1, \dots, m_r)$ if and only if*

$$2g_Z - 2 = |G| \left[2g - 2 + \sum_{i=1}^{k} \left(1 - \frac{1}{m_i} \right) \right]$$

and G has a generating $(g; m_1, \dots, m_r)$-vector.

We now give an application of the preceding theorem to the existence of holomorphic maps with prescribed ramification data (or, equivalently, ramification divisor).

Theorem I.13. *Let Y be a compact Riemann surface of genus g, and let $\{y_1, \dots, y_r\}$ be a finite subset of Y. Fix a transitive subgroup G of $\mathfrak{S}_d$. The following statements are equivalent*:

(1) *There is a compact Riemann surface X and a holomorphic map $f\colon X \to Y$ of degree d and branch locus $\{y_1, \dots, y_r\}$ such that* $\mathrm{Mon}(f) = G$ *and y_i is of type $[\nu_{i,1}, \dots, \nu_{i,k_i}]$ for each $i \in \{1, \dots, r\}$.*

(2) *The group G has a generating $(g; m_1, \dots, m_r)$-vector*

$$(a_1, \dots, a_g, b_1, \dots, b_g, c_1, \dots, c_r)$$

such that c_i has cycle structure $[\nu_{i,1}, \dots, \nu_{i,k_i}]$ with $m_i = \mathrm{lcm}\{\, \nu_{i,j} : 1 \le j \le k_i \,\}$ for each $i = 1, \dots, r$.

Proof. We first prove that item (2) implies item (1), so let us assume that item (2) is fulfilled. Theorem I.12 implies the existence of a compact Riemann surface Z where G acts with signature $(g; m_1, \dots, m_r)$. According to [**4**, Equations (2.2) and (2.5)], the elements $c_i \in G$ are the corresponding images of torsion elements $\gamma_1, \dots, \gamma_r$ of a group G^* that acts on the uniformization of Z, the action that induces that of G on Z. The elements γ_i generate all the torsion groups of G^* up to conjugacy, hence the cyclic groups $\langle \gamma_i \rangle$ (and their conjugates) are the only subgroups of G^* that fix points; correspondingly, the subgroups $\langle c_i \rangle$ of G (and their conjugates) are the stabilizers of the ramification points of the quotient map π_G. Therefore, if we denote by C_i the conjugacy class of $\langle c_i \rangle$ in G, then the geometric signature of G is $(g; C_1, \dots, C_r)$.

Set $H := \mathrm{Stab}_G(1)$. According to Theorem I.4, the induced map $\pi^H\colon W/H \to W/G$ satisfy $\mathrm{Mon}(\pi^H) = G$ and $\deg \pi^H = |H| = d$. Moreover, since π_G is the Galois closure of π^H, the isomorphism of Theorem I.5 becomes an equality. Set $p_i \in Z$ such that $\mathrm{Stab}_G(p_i) = \langle c_i \rangle$, and denote $q_i := \pi_G(p_i)$ for each $i = 1, \dots, r$. Theorem I.10 implies that there is a small loop α_i around q_i such that $\mathrm{Stab}_G(p_i) = \langle \rho(\alpha_i) \rangle$ for each $i = 1, \dots, r$, where $\rho\colon \pi_1(Z/G - \{q_1, \dots, q_r\}) \to \mathfrak{S}_d$ is the monodromy representation of π^H. Moreover, since both c_i and $\rho(\alpha_i)$ generate the same cyclic permutation group, they have the same cycle structure, namely $[\nu_{i,1}, \dots, \nu_{i,k_i}]$; thereby, Theorem I.8 implies that if we set $X := Z/H$ and $f := \pi^H$, then item (1) is fulfilled.

Now we prove that item (1) implies item (2). Assume item (1), then, according to Theorem I.5, the Galois closure of f, namely $\hat{f}\colon \hat{X} \to Y$, verifies that $\mathrm{Aut}(\hat{f}) \cong G$. So G acts on $\hat{X}$ and each nontrivial stabilizer correspond to a branch value y_i. According to Theorem I.10, those stabilizers, namely S_i, are respectively generated by elements with cycle structure $[\nu_{i,1}, \dots, \nu_{i,k_i}]$, hence of order $m_i = \mathrm{lcm}\{\, \nu_{i,j} : j = 1, \dots, k_i \,\}$; therefore, by Theorem I.12, the group G has a generating $(g; m_1, \dots, m_r)$-vector

$$(a_1, \dots, a_g, b_1, \dots, b_g, c_1, \dots, c_r).$$

Moreover, the elements c_i are generators of the stabilizers S_i, and hence of cycle structure $[\nu_{i,1}, \dots, \nu_{i,k_i}]$. □

Corollary I.14. *Consider a covering map $f\colon X \to Y$ between compact Riemann surfaces. Let $\hat{f}\colon \hat{X} \to Y$ be its Galois closure. Suppose the branch values of f are of types $t_1, \dots, t_k$ and there are exactly n_i branch values of type t_i for each $i \in \{1, \dots, k\}$; then the geometric* *signature of the action of* $\mathrm{Mon}(f)$ *on* $\hat{X}$ *is*

$$(g_Y; C_{1,1}, \dots, C_{1,n_1}, \dots, C_{k,1}, \dots, C_{k,n_k}),$$

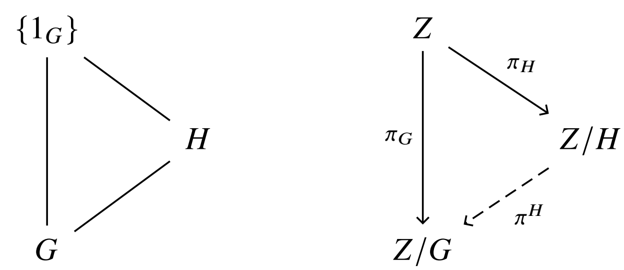
where $C_{i,j}$ *is a cyclic group generated by a permutation of type* t_i *for each* $i \in \{1, \dots, k\}$ *and* $j \in \{1, \dots, n_k\}$*.*

Proof. Follows directly from item (1) implying item (2) in Theorem I.13, considering that conjugacy classes in the geometric signature of $\mathrm{Mon}(f)$ on $\hat{X}$ are in one to one relation with the c_i permutations in the generating $(g; m_1, \dots, m_r)$-vector of item (2) of the theorem. □

4. Intermediate coverings of a Galois covering

In this section, we give some useful results regarding intermediate coverings of a Galois covering; more precisely, we give computations, which can be found at [**23**, Chapter 3], of the ramification data of the intermediate coverings and the genera of the respective intermediate covers in group-theoretic terms.

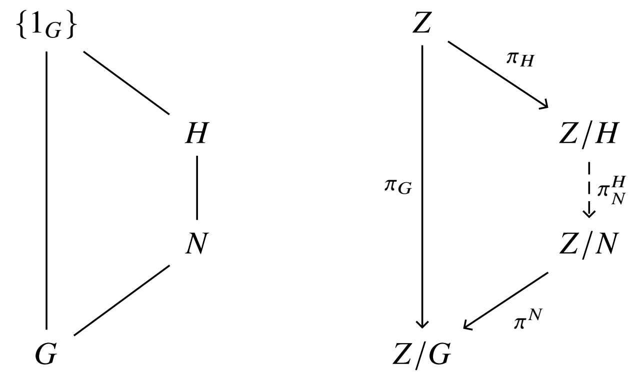
Consider a finite group G acting holomorphically and effectively on compact Riemann surface Z and denote the quotient map by $\pi_G \colon Z \to Z/G$; also let B denote the branch locus of π_G. For each subgroup H of G there is a quotient map $\pi_H \colon Z \to Z/H$, which also induces a (non-necessarily Galois) holomorphic map $\pi^H \colon Z/H \to Z/G \colon [z]_H \mapsto [z]_G$ such that the following diagram (on the right side) commutes:



According to Theorem I.6, we must have $\mathrm{Mon}(\pi_G) \cong G$; therefore, since the correspondence in Theorem I.2 is one-to-one and order reversing, every intermediate covering of π_G is given as above; compare with equation (I.1). If we denote the isomorphic copy of H into $\mathrm{Mon}(\pi_G)$ by $H_{\mathrm{Mon}(\pi_G)}$ and the monodromy representation of π_G by ρ, then equation (I.2) and [**22**, 1.4.6] imply the following sequence of isomorphisms

$$\mathrm{Aut}(\pi^H) \cong \frac{\mathrm{N}_{\rho^{-1}\mathrm{Mon}(\pi_G)}\left(\rho^{-1}H_{\mathrm{Mon}(\pi_G)}\right)}{\rho^{-1}H_{\mathrm{Mon}(\pi_G)}} \cong \frac{\mathrm{N}_{\mathrm{Mon}(\pi_G)}\left(H_{\mathrm{Mon}(\pi_G)}\right)}{H_{\mathrm{Mon}(\pi_G)}} \cong \frac{\mathrm{N}_G(H)}{H}. \tag{I.3}$$

Hence we can determine if the induced map π^H is Galois in terms of just H and G. Also, if there are two subgroups H and N of G with $H \subset N$, we denote by π_N^H the intermediate covering of π_N induced by the action of H on Z as in the following diagram:



We summarize some direct consequences of these correspondences as follows.

Proposition I.15. *Given a Galois covering $\pi_G \colon Z \to Z/G$ and subgroups H and N of G with $H \subset N$, we have that*:

(1) *The maps π_H and π^H have intermediate coverings if and only if there is a subgroup K of G with $H \subsetneq K \subsetneq G$ and $\{1_G\} \subsetneq K \subsetneq H$, respectively.*
(2) *The map π_H is Galois.*
(3) *The maps π^H and π_N^H are Galois if and only if H is normal in G and H is normal in N, respectively*; *in the case these maps are Galois, they are given by the action of G/H and N/H, respectively.*

If we know the genus $g_{Z/G}$, then Theorem I.11 yields the genus of Z; the following theorem gives a manner to compute the genus $g_{Z/H}$ of each intermediate covering of π_G in terms of its geometric signature.

Proposition I.16 ([**23**, Proposition 3.2.3])**.** *Let Z be a curve with G action and geometric signature $(g; C_1, \ldots, C_r)$. For each subgroup H of G, the genus of Z/H is given by*

$$g_{Z/H} = [G:H](g-1) + 1 + \frac{1}{2}\sum_{i=1}^{r}\sum_{l\in\Omega_i} \frac{[\mathrm{N}_G(G_i):G_i]|G_i^l \cap H|}{|H|}\left(\frac{|G_i^l|}{|G_i^l \cap H|} - 1\right),$$

where G_i is a representative of C_i and Ω_i is a transversal of the normalizer of G_i in G for each $i = 1, \ldots, r$.

In order to determine the geometric signature of the quotient π_H we just need the stabilizer of each ramification point; them can be easily obtained from $\mathrm{Stab}_H(p) = \mathrm{Stab}_G(p) \cap H$.

The following theorem gives a manner to compute the ramification data of π^H in terms of H and the geometric signature of the action of G.

Proposition I.17 ([**23**, Proposition 3.2.5])**.** *Let Z be a curve with G action and geometric signature $(g; C_1, \ldots, C_r)$ and H a subgroup of G. If G_i is a representative of C_i and $p \in Z/G$*

is the image of the points of Z associated to C_i, then there are $[\mathrm{N}_G(G_i) : G_i]|G_i^l \cap H|/|H|$ *points above p of multiplicity* $|G_i|/|G_i^l \cap H|$ *for each l in a transversal of* $\mathrm{N}_G(G_i)$ *in G.*

Thereby, the ramification data of each intermediate covering and the genera of their respective Riemann surfaces can be computed in a group-theoretical manner in terms of the geometric signature of the action of G on Z. All these computations were implemented into a SageMath [**26**] class, namely `GaloisCovering`, using mostly GAP [**11**] functions, its source code is given in appendix A and `https://bit.ly/3wDmLiL`. There is also a previous (although non object-oriented) implementation through GAP [**11**], which can be found in [**23**, appendix A].

CHAPTER II

Representation Theory

In this chapter we will introduce some results on representation theory that will be needed and fix the corresponding notation. Throughout this chapter G denotes a finite group. For any field k, we will denote the set of *irreducible representations* of G over k modulo isomorphism, as defined in [**8**, (10.2) Definition], by $\mathrm{Irr}_k(G)$. We will only consider fields of characteristic 0.

1. Complex and rational irreducible representations

As it is usually done, a *complex representation* $\rho\colon G \to \mathrm{GL}(V)$ will be denoted just by V, its *character* by χ_V and its *Schur index* by m_V (see [**8**, (30.3) Definition and (41.4) Definition] for the respective definitions). As stated in [**9**, Proposition 2.30], the number of irreducible complex representations of G modulo isomorphism is equal to the number of *conjugacy classes* of G. Moreover, if $\mathbb{C}_{\mathrm{class}}(G)$ denotes the space of *class functions* on G, its irreducible characters form an orthonormal basis for $\mathbb{C}_{\mathrm{class}}(G)$ with respect to the usual inner product

$$\langle \chi, \xi \rangle_G = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \xi(g)$$

for $\chi, \xi \in \mathbb{C}_{\mathrm{class}}(G)$. Let $\mathrm{Irr}_{\mathbb{C}}(G) = \{V_1, \ldots, V_n\}$ with V_1 the *trivial representation*. Choose a complete set of representatives $\{c_1, \ldots, c_n\}$ of the conjugacy classes of G; without loss of generality, we can set $c_1 = \mathrm{Id}_G$. Let a_j be the cardinality of the conjugacy class of c_j for each $j \in \{1, \ldots, n\}$. Then:

- $\chi_{V_1}(c_j) = 1$ for each $j \in \{1, \ldots, n\}$, and
- $\chi_{V_j}(\mathrm{Id}_G) = \dim V_j$ for each $j \in \{1, \ldots, n\}$.

We call Table 1 the *character table* of G (see [**9**, p. 14]).

Because in chapter V we deal with group actions on Jacobian varieties, we are more interested in $\mathrm{Irr}_{\mathbb{Q}}(G)$ than in $\mathrm{Irr}_{\mathbb{C}}(G)$. According to [**8**, Theorem 70.12], for each $V \in \mathrm{Irr}_{\mathbb{C}}(G)$ there is a minimal algebraic extension $\mathbb{Q}[V]/\mathbb{Q}$ in which V is realizable and such that $[\mathbb{Q}[V] : \mathbb{Q}[\chi_V]] = m_V$, where $\mathbb{Q}[\chi_V]$ denotes the extension of $\mathbb{Q}$ by the values of χ_V. The fields $\mathbb{Q}[V]$ and $\mathbb{Q}[\chi_V]$ are called *definition field* and *field of characters* of V, respectively.

Now let $\mathrm{Gal}(k'/k)$ denote the *Galois group* of a field extension k'/k (see [**13**, p. 252]). For each $V \in \mathrm{Irr}_{\mathbb{C}}(G)$ and $\sigma \in \mathrm{Gal}(\mathbb{Q}[V]/\mathbb{Q})$, we denote by V^σ the *conjugate representation* of V by σ as defined on [**8**, p. 471]; the representation V^σ is also in $\mathrm{Irr}_{\mathbb{C}}(G)$ and both

Table 1. Character table of a group G

	1	a_2		a_n
G	Id_G	c_2	$\dots$	c_n
V_1	1	1	$\dots$	1
V_2	$\dim(V_2)$	$\chi_{V_2}(c_2)$	$\dots$	$\chi_{V_2}(c_n)$
		$\vdots$		
V_n	$\dim(V_n)$	$\chi_{V_n}(c_2)$	$\dots$	$\chi_{V_n}(c_n)$

representations, V and V^σ, share the same definition field. Moreover, they are isomorphic if and only if $\sigma \in \mathrm{Gal}(\mathbb{Q}[V]/\mathbb{Q}[\chi_V])$. As stated in [**8**, (70.15) Theorem], for each rational irreducible representation W, there is a complex irreducible representation V such that

$$W \otimes \mathbb{C} \cong \bigoplus_{\sigma} m_V V^\sigma,$$

where σ runs over $\mathrm{Gal}(\mathbb{Q}[\chi_V]/\mathbb{Q})$; we say that V is *Galois associated* to W. We also have that $\dim_{\mathbb{Q}} W = m_V \cdot |\mathrm{Gal}(\mathbb{Q}[\chi_V/\mathbb{Q}])| \cdot \dim_{\mathbb{C}} V$ and

$$\chi_W = m_V \sum_{\sigma} \chi_V^\sigma, \tag{II.1}$$

where $\chi_V^\sigma = \sigma \circ \chi_V$ for each $\sigma \in \mathrm{Gal}(\mathbb{Q}[\chi_V]/\mathbb{Q})$. We will usually refer to the rational representation W and its complex realization $W \otimes \mathbb{C}$ interchangeably.

According to [**25**, Corollary 1 on p. 103], the number of rational irreducible representations of G is equal to the number of conjugacy classes of its cyclic subgroups; this motivates the following definition.

Definition II.1 (Rational conjugacy classes). Two elements $x, y \in G$ are *rational conjugates* if $\langle x \rangle = \langle y \rangle^z$ for some $z \in G$. Rational conjugacy is a equivalence relation and each class is called a *rational conjugacy class*.

Each rational character is constant in each rational conjugacy class (see [**25**, Theorem 29 part (b)]). Suppose that:

- We have $\mathrm{Irr}_{\mathbb{Q}}(G) = \{W_1, \dots, W_r\}$ with W_1 trivial.
- The set $\{\tilde{c}_1, \dots, \tilde{c}_r\}$ is a complete set of representatives of rational conjugacy classes of G with $\tilde{c}_1 = \mathrm{Id}_G$.
- The class of each $\tilde{c}_i$ has exactly $\tilde{a}_i$ elements.

Then we can compute Table 2, called the *rational character table* of G, from Table 1 and equation (II.1).

There is a unique decomposition

$$\mathbb{Q}[G] = \mathbb{Q}[G]e_1 \oplus \cdots \oplus \mathbb{Q}[G]e_r \tag{II.2}$$

TABLE 2. Rational character table of a group G

G	1 Id_G	$\tilde{a}_2$ $\tilde{c}_2$	$\dots$	$\tilde{a}_r$ $\tilde{c}_r$
W_1	1	1	$\dots$	1
W_2	$\dim(W_2)$	$\chi_{W_2}(\tilde{c}_2)$	$\dots$	$\chi_{W_2}(\tilde{c}_r)$
		$\vdots$		
W_r	$\dim(W_r)$	$\chi_{W_r}(\tilde{c}_2)$	$\dots$	$\chi_{W_r}(\tilde{c}_r)$

of the *group algebra* $\mathbb{Q}[G]$ into simple $\mathbb{Q}[G]$-algebras where the e_i are mutually orthogonal central idempotents respectively associated to the irreducible rational representations W_i of G (see [**8**, §33]). Moreover, according to [**2**, equation (13.6) on p. 433], if $V_j \in \mathrm{Irr}_{\mathbb{C}}(G)$ is Galois associated to W_j, then

$$e_j = \frac{\dim V_j}{|G|} \sum_{g \in G} \mathrm{tr}\big(\chi_{V_j}(g)\big) g,$$

where $\mathrm{tr}\big(\chi_{V_j}(g)\big)$ denotes the trace of $\chi_{V_j}(g)$ viewed as an element of $\mathrm{Gal}(\mathbb{Q}[\chi_{V_j}]/\mathbb{Q})$. Equation (II.2) is called the *rational isotypical decomposition* of $\mathbb{Q}[G]$. Furthermore, for each e_j there are primitive orthogonal idempotents $q_{j,1}, \dots, q_{j,l_j} \in \mathbb{Q}[G]e_j$ such that

$$\mathbb{Q}[G]e_j = \mathbb{Q}[G]q_{j,1} \oplus \cdots \oplus \mathbb{Q}[G]q_{j,l_j} \tag{II.3}$$

is a decomposition of $\mathbb{Q}[G]e_j$ into minimal right ideals (see [**2**, equation (13.8)]). This decomposition is not unique but, according to Wedderburn's theorem (see [**8**, (26.8)]), the module $\mathbb{Q}[G]e_j$ is isomorphic to $\mathrm{End}\Big(D_j^{l_j}\Big)$, where D_j is the (uniquely determined up to isomorphism) skew-field $q_j\mathbb{Q}[G]q_j$ for any primitive idempotent q_j of $\mathbb{Q}[G]e_j$; so l_j is uniquely determined. Moreover, according to [**7**, equation (2.4)],

$$l_j = \frac{\dim V_j}{m_{V_j}}; \tag{II.4}$$

note that m_{V_j} depends only on W_j (so we can denote the Schur index just by m_{W_j}), and that $|\mathrm{Gal}(\mathbb{Q}[\chi_{V_j}/\mathbb{Q}])|$ is the number of complex irreducible representations Galois associated to the rational representation W_j. In [**7**, Theorem 3.3, Corollary 3.5 and Corollary 3.6], a method for explicitly constructing the primitive idempotents $q_{j,i}$ is given. The (non-unique) decomposition

$$\mathbb{Q}[G] = \bigoplus_{j=1}^{r} \bigoplus_{i=1}^{l_j} \mathbb{Q}[G]q_{j,i} \tag{II.5}$$

will be called *group algebra decomposition*.

2. Representations induced by a trivial representation

In this section, we study representations of G *induced* by the trivial representation of a subgroup H. Chapter V deals with *Prym varieties* and, in order to decompose the *Jacobian* variety of a Riemann surface into Prym varieties, the following type of representations will be especially useful.

For a subgroup H of G, we set

$$\rho_H := \mathrm{Ind}_H^G(1_H),$$

where 1_H denotes the trivial representation of H and $\mathrm{Ind}_H^G(W)$ the G-representation induced by an H-representation W (see [**25**, subsection 3.3 and Chapter 7]); since 1_H is rational, so it is ρ_H. Let $\mathrm{Ind}_H^G(\mu)$ and $\mathrm{Res}_H^G(\nu)$ denote the induced and restricted *class functions* (see [**25**, subsection 7.2]) of $\mu \in \mathbb{C}_{\mathrm{class}}(H)$ and $\nu \in \mathbb{C}_{\mathrm{class}}(G)$, respectively. If we denote the character of ρ_H by χ_H, then Frobenius reciprocity (see [**25**, Theorem 13]) yields that

$$\langle \chi_H, \psi \rangle_G = \left\langle 1_H, \mathrm{Res}_H^G \psi \right\rangle_H = \frac{1}{|H|} \sum_{h \in H} \psi(h)$$

for each G-character ψ; in particular, for any $W \in \mathrm{Irr}_{\mathbb{Q}}(G)$ we have

$$\langle \chi_H, \chi_W \rangle_G = \frac{1}{|H|} \sum_{h \in H} \chi_W(h). \tag{II.6}$$

Thus, we can easily compute the rational irreducible components of ρ_H from the rational character table of G and H. As we summarize in the following result, we only need the data about the rational conjugacy classes of H and the values of the irreducible rational G-representations on those classes.

Proposition II.1. *Let G be a finite group. If H is a subgroup of G with rational character table as in Table 2, then*

$$\langle \chi_H, \chi_W \rangle_G = \frac{1}{|H|} \sum_{i=1}^{r} \chi_W(\tilde{c}_i)\tilde{a}_i$$

for all $W \in \mathrm{Irr}_{\mathbb{Q}}(G)$.

And, directly from the proposition above, we get the following result.

Corollary II.2. *Let G be a finite group. If H is a subgroup of G with rational character table as in Table 2 and $\mathrm{Irr}_{\mathbb{Q}}(G) = \{W_1, \ldots, W_{r_G}\}$, then*

$$\rho_H = \frac{1}{|H|} \sum_{j=1}^{r_G} \left(\sum_{i=1}^{r} \chi_{W_j}(\tilde{c}_i)\tilde{a}_i \right) W_j.$$

If V is any complex G-representation, a more geometrical characterization of the decomposition above is given by

$$\langle \chi_H, \chi_V \rangle_G = \dim_{\mathbb{C}}(\mathrm{Fix}_H V),$$

where $\operatorname{Fix}_H V$ denotes the V-subspace of points fixed by H. Therefore, if we decompose ρ_H into rational irreducible representations—this is called *rational isotypical decomposition* as in equation (II.2)—then the times that each rational irreducible representation is contained in ρ_H is given by the following result.

Theorem II.3 ([**7**, Lema 4.3])**.** *Set* $\operatorname{Irr}_{\mathbb{Q}}(G) = \{W_1, \dots, W_r\}$ *and for each* W_j *choose a Galois associated representation* $V_j \in \operatorname{Irr}_{\mathbb{C}}(G)$*. Also let* H *be a subgroup of* G*. Then the rational isotypical decomposition of* ρ_H *is given by*

$$\rho_H \cong \bigoplus_{j=1}^{r} \frac{\dim_{\mathbb{C}}(\operatorname{Fix}_H V_j)}{m_V} W_j.$$

CHAPTER III

Isotypical and Group Algebra Decomposition of a Jacobian Variety

For the whole chapter, let X and Y denote two compact Riemann surfaces with a covering map $f\colon X \to Y$ of degree d between them. For every curve Z, we denote its *Jacobian variety* by $(\mathrm{Jac}(Z), \Theta_Z)$ (see its definition and main properties in [**2**, section 11.1]); also, for any *polarized abelian variety* (A, Θ), we denote the natural homomorphism onto its dual by $\phi_\Theta\colon A \to \hat{A}$, we also set $\mathrm{K}(\Theta) := \ker \phi_\Theta$ (see [**2**, pp. 36–37]).

1. Prym variety of a covering map

The pullback $f^*\colon \mathrm{Jac}(Y) \to \mathrm{Jac}(X)$ is a homomorphism with finite kernel; thus an *isogeny* onto its image $f^* \mathrm{Jac}(Y)$. Moreover, we have the following result.

Proposition III.1 ([**2**, Proposition 11.4.3])**.** *The homomorphism f^* is not injective if and only if f factorizes via a cyclic étale covering f' of degree greater than* 2 *as in the following commutative diagram*:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & \searrow \qquad \nearrow_{f'} & \\ & Z & \end{array}$$

In this case, we have that $\ker f'^*$ *is cyclic of order* $\deg f'$.

Recall from [**2**, Theorem 5.3] that, as for any abelian variety, we have a bijective correspondence between:

(1) *abelian subvarieties* A of a Jacobian J; and
(2) *symmetric idempotents* ϵ into the *rational endomorphisms* $\mathrm{End}_{\mathbb{Q}}(J)$.

In virtue of this correspondence, each abelian subvariety A has a unique complementary subvariety: if ϵ is the idempotent associated to A, then its complement is the subvariety associated to $1 - \epsilon$. Moreover, in [**2**, section 5.3] is stated that the *norm* endomorphism $\mathrm{Nm}(A) \in \mathrm{End}(J)$ of J associated to A satisfies $\mathrm{Nm}(A) = e(A)\epsilon$, where $e(A)$ is the *exponent* of A, that is, the exponent of the polarization induced by the inclusion map $A \hookrightarrow J$.

Definition III.1. The *Prym variety* of a covering map $f\colon X \to Y$ between compact Riemann surfaces, denoted by $\mathrm{Prym}(f)$, is the complement of $f^* \mathrm{Jac}(Y)$ in $\mathrm{Jac}(X)$.

With respect to the polarization induced in $\mathrm{Prym}(f)$ by Θ_X, we have the following result.

Theorem III.2 ([**21**, Theorem 2.5])**.** *Let $f\colon X \to Y$ be a covering map of degree d. Denote by $\Theta_{f^*\operatorname{Jac}(Y)}$ and by $\Theta_{\operatorname{Prym}(f)}$ the polarizations induced by Θ_X in $f^*\operatorname{Jac}(Y)$ and $\operatorname{Prym}(f)$, respectively. Then*:

(1) *The pullback of $\Theta_{f^*\operatorname{Jac}(Y)}$ by f^* is analytically equivalent to $\Theta_Y^{\otimes d}$, and $\ker f^*$ is an isotropic subgroup of $\operatorname{Jac}(Y)[d]$ with respect to the Weil form associated to $\Theta_Y^{\otimes d}$.*

(2) *The homomorphism f^* induces an isomorphism*

$$f^*\colon \frac{(\ker f^*)^\perp}{\ker f^*} \to \operatorname{K}(\Theta_{f^*\operatorname{Jac}(Y)}),$$

where orthogonality is with respect to the Weil form associated to $\Theta_Y^{\otimes d}$. Moreover, we have $\operatorname{K}(\Theta_{\operatorname{Prym}(f)}) = \operatorname{K}(\Theta_{f^*\operatorname{Jac}(Y)}) = f^*\operatorname{Jac}(Y) \cap \operatorname{Prym}(f)$.

(3) *The homomorphism*

$$\begin{aligned}\mu\colon \operatorname{Jac}(Y) \times \operatorname{Prym}(f) &\to \operatorname{Jac}(X)\\ (y, p) &\mapsto f^*(y) + p\end{aligned}$$

is an isogeny, and the natural projection $\pi_1\colon \operatorname{Jac}(Y)\times\operatorname{Prym}(f) \to \operatorname{Jac}(Y)$ restricts to an isomorphism

$$\pi_1\colon \ker\mu \to (\ker f^*)^\perp.$$

Moreover, as a direct consequence of the previous theorem and [**2**, Corollary 5.3.6] we get the following result.

Proposition III.3. *The map $\phi_{\mu^*\Theta_X}\colon \operatorname{Jac}(Y) \times \operatorname{Prym}(f) \to \widehat{\operatorname{Jac}(Y)} \times \widehat{\operatorname{Prym}(f)}$ satisfies*

$$\phi_{\mu^*\Theta_X} = \begin{pmatrix} \phi_{\Theta_Y^{\otimes d}} & 0 \\ 0 & \phi_{\Theta_{\operatorname{Prym}(f)}} \end{pmatrix}.$$

Consider now a Galois covering map $\pi_G\colon Z \to Y$ given by the quotient of a curve Z by the action of a group G. For each $g \in G$, the pullback $\left(g^{-1}\right)^*\colon \operatorname{Jac}(Z) \to \operatorname{Jac}(Z)$ is an automorphism of $\operatorname{Jac}(Z)$; thereby, since $\left((gh)^{-1}\right)^* = \left(g^{-1}\right)^* \circ \left(h^{-1}\right)^*$, the group G acts also on $\operatorname{Jac}(Z)$. The automorphism $\left(g^{-1}\right)^*$ of $\operatorname{Jac}(Z)$ will be denoted just by g.

The action of G on $\operatorname{Jac}(Z)$ induces a natural algebra homomorphism

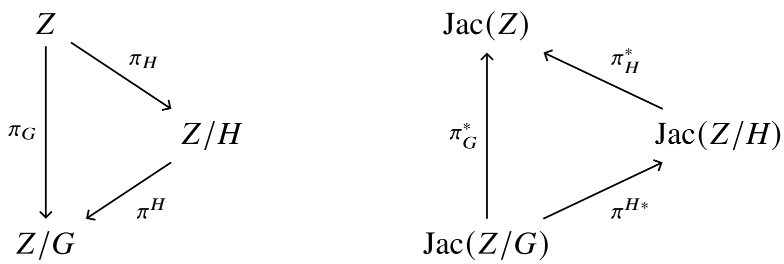
$$\mathbb{Q}[G] \to \operatorname{End}_{\mathbb{Q}}(\operatorname{Jac}(Z));$$

the elements of the image of this homomorphism will be denoted just as elements of $\mathbb{Q}[G]$. The following endomorphism is of particular interest.

Definition III.2. The *norm endomorphism* $\operatorname{Nm}_G \in \operatorname{End}(\operatorname{Jac}(Z))$ is given by $\operatorname{Nm}_G = \sum_{g\in G} g$.

The following theorem summarizes some results about Jacobian and Prym varieties associated to intermediate coverings of Galois map; all of them are proved in [**21**, section 3].

Theorem III.4. *Set a finite group G acting on a curve Z and a subgroup H of G. Denote the canonical quotient maps and their pullbacks as in the following commutative diagrams*:



Then we have:

(1) *The pullback* $\pi_G^*\big(\mathrm{Jac}(Z/G)\big)$ *equals* $\big(\mathrm{Jac}(Z)^G\big)^0$.
(2) *The Prym variety* $\mathrm{Prym}(\pi_G)$ *equals* $\big(\ker(\mathrm{Nm}_G)\big)^0$.
(3) *If* $\{g_1, \ldots, g_r\}$ *is a complete set of representatives for* G/H, *then*

$$\pi_H^*\big(\mathrm{Prym}(\pi^H)\big) = \left\{ z \in \mathrm{Fix}_H(\mathrm{Jac}(Z)) : \sum_{i=1}^{r} g_i(z) = 0 \right\}^0.$$

2. Decomposition of a Jacobian variety into Prym varieties

Throughout this section, assume that $\pi_G \colon Z \longrightarrow Z/G$ is a Galois covering map given by the action of a finite group G on a compact Riemann surface Z. For any $\alpha \in \mathrm{End}_{\mathbb{Q}}(\mathrm{Jac}(Z))$, we define $\mathrm{im}(\alpha) := \mathrm{im}(m\alpha)$ where $m \in \mathbb{Z}^+$ is such that $m\alpha \in \mathrm{End}(\mathrm{Jac}(Z))$. This definition certainly does not depend on m.

If we set $\mathrm{Irr}_{\mathbb{Q}}(G) = \{W_1, \ldots, W_r\}$, then the rational isotypical decomposition of $\mathbb{Q}[G]$ given by equation (II.2) induces a decomposition of $\mathrm{Jac}(Z)$ as follows (the original result is more general; not just for Jacobian varieties, but for any abelian variety).

Proposition III.5 ([**14**, Proposition 1.1])**.**

- *Each abelian subvariety* $\mathrm{im}\, e_i$ *is G-stable with* $\mathrm{Hom}_G(\mathrm{im}\, e_i, \mathrm{im}\, e_j) = 0$ *for* $i \neq j$.
- *The addition map induces an isogeny*

$$\mu \colon\; \mathrm{im}\, e_1 \times \cdots \times \mathrm{im}\, e_r \longrightarrow \mathrm{Jac}(Z).$$

This decomposition is called the *isotypical decomposition* of $\mathrm{Jac}(Z)$ and it is unique up to a permutation of the factors since the idempotents e_i are uniquely determined.

Recall the group algebra decomposition of $\mathbb{Q}[G]$, given by equation (II.5). As stated in [**2**, p. 434], the abelian subvarieties $\mathrm{im}\, q_{j,1}, \ldots, \mathrm{im}\, q_{j,l_j}$ are pairwise isogenous for each fixed $j = 1, \ldots, r$. Thereby, for each $j = 1, \ldots, r$, there is an abelian subvariety A such that $\mathrm{im}\, e_j$ is isogenous to A^{l_j} (we can set $A := \mathrm{im}\, q_{j,1}$ for example, but this certainly does not determine A uniquely, since the $q_{j,i}$ are not uniquely determined by G). Thereby, we get the following result, which is just [**14**, Theorem 2.2] applied to Jacobian varieties.

Theorem III.6. *Let G be a finite group acting on a Jacobian variety* $\mathrm{Jac}(Z)$. *Suppose* $\mathrm{Irr}_{\mathbb{Q}}(G) = \{W_1, \ldots, W_r\}$ *and set* $l_j := \dim_{D_j}(W_j)$ *as in equation* (*II*.4). *Then there are abelian subvarieties* $A_1, \ldots, A_r$ *of* $\mathrm{Jac}(Z)$ *and an isogeny*

$$\mathrm{Jac}(Z) \sim A_1^{l_1} \times \cdots \times A_r^{l_r}.$$

This is called the *group algebra decomposition* of $\mathrm{Jac}(Z)$ with respect to G. We are interested in describing the abelian subvarieties A_i as Jacobian or Prym varieties of intermediate coverings of f; namely, Galois covering maps $\pi_H \colon Z \to Z/H$ and induced covering maps of the form $\pi^H \colon Z/H \to Z/G$ or $\pi_H^K \colon Z/K \to Z/H$, where H and K are subgroups of G (see section 4 of chapter I). We set W_1 as the trivial representation so $l_1 = 1$ and, according to item (1) of Theorem III.4, we have $A_1 \sim \mathrm{Jac}(Z/G)$. For the rest of the abelian subvarieties A_i we have the following results.

Theorem III.7 ([**7**, Proposition 5.2])**.** *Given a Galois covering $\pi_G \colon Z \to Z/G$, consider the group algebra decomposition of* $\mathrm{Jac}(Z)$ *given by*

$$\mathrm{Jac}(Z) \sim \mathrm{Jac}(Z/G) \times A_2^{l_2} \times \cdots \times A_r^{l_r},$$

where $l_j = \dim(V_j)/m_j$ for a complex irreducible representation V_j Galois associated to W_j of Schur index m_j. If H is a subgroup of G, then the isotypical decomposition of $\mathrm{Jac}(Z/H)$ *is given by*

$$\mathrm{Jac}(Z/H) \sim \mathrm{Jac}(Z/G) \times A_2^{s_2} \times \cdots \times A_r^{s_r},$$

where $s_j = \dim_{\mathbb{C}}(\mathrm{Fix}_H V_j)/m_j$.

Corollary III.8 ([**7**, Corollary 5.4])**.** *Given a Galois covering $\pi_G \colon Z \to Z/G$, consider the group algebra decomposition of* $\mathrm{Jac}(Z)$ *given as*

$$\mathrm{Jac}(Z) \sim \mathrm{Jac}(Z/G) \times A_2^{l_2} \times \cdots \times A_r^{l_r}.$$

Then, for any subgroups H and N of G such that $H \subset N$, the corresponding decomposition of $\mathrm{Prym}(\pi_H^N)$ *is given as follows*:

$$\mathrm{Prym}(\pi_H^N) \sim A_2^{t_2} \times \cdots \times A_r^{t_r},$$

where

$$t_j = \frac{\dim_{\mathbb{C}}(\mathrm{Fix}_H V_j)}{m_j} - \frac{\dim_{\mathbb{C}}(\mathrm{Fix}_N V_j)}{m_j}.$$

Remark III.1. In [**7**], is also established that the isogenies above are inclusions and pullbacks like π_G^* and π_H^*.

The corollary below will be very useful; it directly follows from Theorem II.3 and Theorem III.8.

Corollary III.9 ([**7**, Corollary 5.6])**.** *Given a Galois covering $\pi_G \colon Z \to Z/G$, consider the group algebra decomposition of* $\mathrm{Jac}(Z)$ *given as*

$$\mathrm{Jac}(Z) \sim \mathrm{Jac}(Z/G) \times A_2^{l_2} \times \cdots \times A_r^{l_r}.$$

Let H and N be subgroups of G with $H \subset N$. If $\rho_H = W_i \oplus \rho_N$ for a rational irreducible representation W_i; then $\operatorname{Prym}(\pi_H^N) \sim A_i$. *Conversely, if* $\operatorname{Prym}(\pi_H^N) \sim A_i$, *then $\rho_H = W_i \oplus \rho_N$.*

3. Prym variety of pairs of coverings

As we will see in Theorem V.8 and is remarked in [**15**], for some Galois coverings $\pi_G \colon Z \to Z/G$ there are components of the group algebra decomposition of $\operatorname{Jac}(Z)$ that are not the Prym variety of any intermediate covering of π_G; however, there is another type of abelian variety associated to the action of G on Z that, in cases as Theorem V.8 (where $G \cong \mathfrak{S}_5$), acts as the missing pieces in the group algebra decomposition of Theorem III.6. In this section, we define that kind of abelian variety; namely, a *Prym variety of pairs of coverings*.

Consider holomorphic maps between compact Riemann surfaces as in the following commutative diagram:

$$\begin{array}{ccccc} & & X & & \\ & {\scriptstyle f_1}\swarrow & & \searrow{\scriptstyle f_2} & \\ X_1 & & & & X_2 \\ & {\scriptstyle g_1}\searrow & & \swarrow{\scriptstyle g_2} & \\ & & Y & & \end{array} \tag{III.1}$$

According to [**15**, Proposition 2.2], if Y is *minimal* (that is, g_1 and g_2 do not both factorize via the same morphism of degree greater or equal than 2), then $f_2^* \operatorname{Prym}(g_2)$ is an abelian subvariety of $\operatorname{Prym}(f_1)$.

Definition III.3. The complementary abelian variety $\operatorname{Prym}(f_1, f_2)$ of $f_2^* \operatorname{Prym}(g_2)$ in $\operatorname{Prym}(f_1)$ with respect to the induced polarization $\Theta_{\operatorname{Prym}(f_1)}$ is called the *Prym variety of the pair of coverings* (f_1, f_2).

As stated in [**15**, Proposition 2.4], we have that $\operatorname{Prym}(f_1, f_2)$ equals $\operatorname{Prym}(f_2, f_1)$ as polarized abelian varieties. Moreover, the Prym variety of a pair of coverings is well-defined for any pair (f_1, f_2) of coverings with the same domain (that is, the curve Y and the maps g_1 and g_2 of the commutative diagram (III.1) are naturally given by the pair (f_1, f_2)); also, we can assume X is *minimal* (that is, f_1 and f_2 do not both factorize via the same morphism of degree greater or equal than 2) by, if necessary, redefining $X = X_1 \times_Y X_2$ (see [**15**, p. 377]). With respect to the dimension of $\operatorname{Prym}(f_1, f_2)$, when X is minimal, the following result is known.

Proposition III.10 ([**15**, Proposition 2.5])**.** *For any pair of coverings as in diagram* (III.1) *where X and Y are minimal, we have*

$$\dim \operatorname{Prym}(f_1, f_2) = (d_1 - 1)(d_2 - 1)(g_Y - 1) + \frac{1}{2}\left[\deg(R_{f_1}) + (d_1 - 1)\deg(R_{g_1}) - \deg(R_{g_2})\right],$$

where d_1 and d_2 are the degrees of f_1 and f_2, respectively.

Back into the context of Galois coverings, let N_1 and N_2 be two subgroups of a group G acting on a compact Riemann surface Z. If we define $M := N_1 \cap N_2$ and $N := \langle N_1, N_2 \rangle$, then the coverings Z/M and Z/N in the following diagram are both minimal:

$$\begin{array}{ccccc} & & Z/M & & \\ & {\scriptstyle \pi^M_{N_1}} \swarrow & & \searrow {\scriptstyle \pi^M_{N_2}} & \\ Z/N_1 & & & & Z/N_2 \\ & {\scriptstyle \pi^{N_1}_N} \searrow & & \swarrow {\scriptstyle \pi^{N_2}_N} & \\ & & Z/N & & \end{array} \tag{III.2}$$

The following result is a generalization of [**15**, Proposition 3.4] for groups where not necessarily every irreducible $\mathbb{Q}$-representation is absolutely irreducible; it relates the group algebra decomposition of the Jacobian variety of a Galois covering with Prym varieties of pairs of its subcoverings.

Theorem III.11. *In the notation of Theorem III.*6 *and diagram* (III.2)*, we have* $\mathrm{Prym}(\pi^M_{N_1}, \pi^M_{N_2}) \sim A_2^{t_2} \times \cdots \times A_r^{t_r}$ *with*

$$t_j = \frac{\dim_{\mathbb{C}}(\mathrm{Fix}_M V_j)}{m_j} + \frac{\dim_{\mathbb{C}}(\mathrm{Fix}_N V_j)}{m_j} - \frac{\dim_{\mathbb{C}}(\mathrm{Fix}_{N_1} V_j)}{m_j} - \frac{\dim_{\mathbb{C}}(\mathrm{Fix}_{N_2} V_j)}{m_j}$$

for $j \in \{1, \ldots, r\}$.

Proof. We have that

$$\mathrm{Jac}(Z/M) \sim \mathrm{Jac}(Z/N) \times \mathrm{Prym}(\pi^{N_2}_N) \times \mathrm{Prym}(\pi^M_{N_2})$$

and

$$\mathrm{Jac}(Z/M) \sim \mathrm{Jac}(Z/N) \times \mathrm{Prym}(\pi^{N_1}_N) \times \mathrm{Prym}(\pi^M_{N_1}).$$

Hence, by Poincare's complete reducibility theorem, we have

$$\mathrm{Prym}(\pi^{N_2}_N) \times \mathrm{Prym}(\pi^M_{N_2}) \sim \mathrm{Jac}(Z/N) \times \mathrm{Prym}(\pi^{N_1}_N) \times \mathrm{Prym}(\pi^M_{N_1});$$

but $\mathrm{Prym}(\pi^M_{N_1}) \sim \mathrm{Prym}(\pi^{N_2}_N) \times \mathrm{Prym}(\pi^{N_1}_N, \pi^{N_2}_N)$, so

$$\mathrm{Prym}(\pi^{N_2}_N) \times \mathrm{Prym}(\pi^M_{N_2}) \sim \mathrm{Prym}(\pi^{N_1}_N) \times \mathrm{Prym}(\pi^{N_2}_N) \times \mathrm{Prym}(\pi^{N_1}_N, \pi^{N_2}_N),$$

and, again by Poincare's complete reducibility theorem, we get

$$\mathrm{Prym}(\pi^M_{N_2}) \sim \mathrm{Prym}(\pi^{N_1}_N) \times \mathrm{Prym}(\pi^{N_1}_N, \pi^{N_2}_N).$$

Theorem III.8 states that $\mathrm{Prym}(\pi^M_{N_2}) \sim A_2^{t_{2,1}} \times \cdots \times A_r^{t_{r,1}}$ and $\mathrm{Prym}(\pi^{N_1}_N) \sim A_2^{t_{2,2}} \times \cdots \times A_r^{t_{r,2}}$ with

$$t_{j,1} = \frac{\dim_{\mathbb{C}}(\mathrm{Fix}_M V_j)}{m_j} - \frac{\dim_{\mathbb{C}}(\mathrm{Fix}_{N_2} V_j)}{m_j}$$

and

$$t_{j,2} = \frac{\dim_{\mathbb{C}}(\mathrm{Fix}_{N_1} V_j)}{m_j} - \frac{\dim_{\mathbb{C}}(\mathrm{Fix}_{N} V_j)}{m_j}.$$

Therefore, once again by Poincare's complete reducibility theorem, we get that $\mathrm{Prym}(\pi_N^{N_1}, \pi_N^{N_2}) \sim A_2^{t_2} \times \cdots \times A_t^{t_r}$ with $t_j = t_{j,1} - t_{j,2}$, and that last equation is equivalent to

$$t_j = \frac{\dim_{\mathbb{C}}(\mathrm{Fix}_{M} V_j)}{m_j} + \frac{\dim_{\mathbb{C}}(\mathrm{Fix}_{N} V_j)}{m_j} - \frac{\dim_{\mathbb{C}}(\mathrm{Fix}_{N_1} V_j)}{m_j} - \frac{\dim_{\mathbb{C}}(\mathrm{Fix}_{N_2} V_j)}{m_j}. \qquad \square$$

CHAPTER IV

Galois closure of a fivefold covering

Let $f\colon X \to Y$ denote a holomorphic map of degree 5 between compact Riemann surfaces. In this chapter, we give necessary and sufficient criteria for a potential ramification data of f to be actually realizable. Then, all possible monodromy groups $\mathrm{Mon}(f)$ modulo conjugation in $\mathfrak{S}_5$ are tabulated in terms of realizable values of R_f depending on the value of g_Y. The main results are exposed in Theorems IV.10 and IV.12 of section 2; the first one tabulates all possible monodromy groups $\mathrm{Mon}(f)$ up to conjugacy in terms of the ramification data of f when $g_Y \geq 1$, and the second one deals with the special case where $g_Y = 0$; that is, when $Y \cong \mathbb{P}^1$. The preceding notation is kept for the rest of this chapter.

1. Realizable ramification data of a fivefold covering

Recall from Definition I.4 that the type of a branch value of the covering map f is the cycle structure of a permutation in $\mathfrak{S}_5$; hence, it can be $[5]$, $[4, 1]$, $[3, 2]$, $[3, 1, 1]$, $[2, 2, 1]$ or $[2, 1, 1, 1]$. In this section, we state necessary and sufficient conditions for a tuple $(t_1, \ldots, t_n)$ of these ramification types to be, once we prescribe branch values $y_1, \ldots, y_n$ in Y, the ramification data of a degree 5 covering. This is achieved in Theorems IV.1 and IV.3, which deals, respectively, with the cases where $g_Y \geq 1$ and $g_Y = 0$.

We can classify the possible types of branch values of f into two kinds:

(1) the even ones, namely $[5]$, $[3, 1, 1]$ and $[2, 2, 1]$; and
(2) the odd ones, namely $[4, 1]$, $[3, 2]$ and $[2, 1, 1, 1]$.

As we will see in the following theorem, this coarse classification is enough to enunciate a necessary and sufficient realizability condition for $(t_1, \ldots, t_n)$ when $g_Y > 0$.

Theorem IV.1. *Consider a compact Riemann surface Y with $g_Y \geq 1$. For an arbitrary set $\{y_1, \ldots, y_n\}$ of points in Y, there is a holomorphic map $f\colon X \to Y$ of degree 5 with branch values $y_1, \ldots, y_n$ of types $t_1, \ldots, t_n$, respectively, if and only if there is an even number of odd branch values.*

Proof. First, sufficiency of the hypothesis is proven. Suppose there is a holomorphic map $f\colon X \to Y$ of degree 5 and branch values $y_1, \ldots, y_n$ of type $t_1, \ldots, t_n$, respectively. According to Theorem I.13, the monodromy group $\mathrm{Mon}(f)$ has a generating

TABLE 3. Permutations for a generating vector of a transitive subgroup of $\mathfrak{S}_5$

$\prod_{i=1}^n c_i$	a_1	b_1	$\prod_{i=1}^{g_Y} [a_i, b_i]$
$(1\,2\,3\,4\,5)$	$(1\,3)(2\,5\,4)$	$(1\,3\,5\,2)$	$(1\,5\,4\,3\,2)$
$(1\,2)(3\,4)$	$(1\,3)(2\,4\,5)$	$(1\,2\,4\,5)$	$(1\,2)(3\,4)$
$(1\,2\,3)$	$(1\,5\,2)(3\,4)$	$(1\,3\,5\,4)$	$(1\,3\,2)$

$(g_Y; m_1, \ldots, m_n)$-vector $(a_1, \ldots, a_{g_Y}, b_1, \ldots, b_{g_Y}, c_1, \ldots, c_n)$ such that c_i has cycle structure t_i for each $i \in \{1, \ldots, n\}$. By Definition I.8 we know that

$$\prod_{i=1}^{n} c_i = \left(\prod_{i=1}^{g_Y} [a_i, b_i]\right)^{-1} \in \mathrm{Mon}(f)'. \tag{IV.1}$$

But $\prod_{i=1}^n c_i$ is even only if there is a even number of odd permutations c_i and, since $\mathrm{Mon}(f)' \subset \mathfrak{S}_5' = \mathfrak{A}_5$, the product $\prod_{i=1}^n c_i$ must be an even permutation. Therefore, there is an even number of permutations c_i; that is, there is an even number of odd branch values.

For proving the necessity of the hypothesis, an explicit construction will be given. Prescribe points $y_1, \ldots, y_n$ in Y and ramification types $t_1, \ldots, t_n$ for those points; assume that there is an even number of odd branch values. According to Theorem I.13, giving a holomorphic map $f\colon X \longrightarrow Y$ of degree 5 with branch values $y_1, \ldots, y_n$ of types $t_1, \ldots, t_n$ is equivalent to giving a generating $(g_Y; m_1, \ldots, m_n)$-vector $(a_1, \ldots, a_{g_Y}, b_1, \ldots, b_{g_Y}, c_1, \ldots, c_n)$ of a transitive subgroup, namely $\mathrm{Mon}(f)$, of $\mathfrak{S}_5$ such that c_i has cycle structure t_i for each $i \in \{1, \ldots, n\}$.

For the case where $n = 0$, since $g_Y \geq 1$, we have the generating $(g_Y;)$-vector

$$((1\,2\,3\,4\,5), \underbrace{\mathrm{Id}, \ldots, \mathrm{Id}}_{2g_Y - 1})$$

of the transitive subgroup $\langle(1\,2\,3\,4\,5)\rangle$ of $\mathfrak{S}_5$.

For the case where $n > 0$, set c_i as any permutation with cycle structure t_i. The permutation $\prod_{i=1}^n c_i$ is even because there is an even number of odd permutations c_i, so it is of type $[5]$, $[3, 1, 1]$ or $[2, 2, 1]$; thereby, up to conjugacy in $\mathfrak{S}_5$, it can be assumed that $\prod_{j=1}^n c_i$ is $(1\,2\,3\,4\,5)$, $(1\,2\,3)$ or $(1\,2)(3\,4)$. Set a_i and b_i as the identity permutation for $i \in \{2, \ldots, g_Y\}$ (if $g_Y > 1$), and, depending on the value of $\prod_{i=1}^n c_i$, set a_1 and b_1 as in Table 3. Thereby, we get a generating $(g_Y; m_1, \ldots, m_n)$-vector of a subgroup G of $\mathfrak{S}_5$; we just need to prove that G is transitive, but:

(1) If $\prod_{i=1}^n c_i = (1\,2\,3\,4\,5)$, then $(1\,2\,3\,4\,5) \in G$ and hence G transitive.
(2) If $\prod_{i=1}^n c_i = (1\,2)(3\,4)$, then $a_1 b_1 = (1\,4\,2\,5\,3) \in G$ and hence G is transitive.
(3) If $\prod_{i=1}^n c_i = (1\,2\,3)$, then $a_1 b_1 = (1\,4\,5\,3\,2) \in G$ and hence G is transitive. $\square$

Motivated by Theorem IV.1, a tuple of ramification types $(t_1 \dots, t_n)$ with an even number of odd branch values will be called *even*; thereby, if $g_Y > 0$, then a tuple of potential ramification types is realizable if and only if it is even.

Items (1) to (3) of the proof of Theorem IV.1 also show that, for any even tuple $(t_1 \dots, t_n)$ with $n > 0$, we can choose a generating vector such that $\mathrm{Mon}(f)$ contains an order 5 permutation; also, that choice is made in such a manner that $\mathrm{Mon}(f)$ contains permutations of order 6 and 4 (see columns a_1 and b_1 in Table 3, respectively). Since $\mathrm{lcm}(4, 5, 6) = 60$, we have that 60 divides $|\mathrm{Mon}(f)|$; hence $\mathrm{Mon}(f) \in \{\mathfrak{A}_5, \mathfrak{S}_5\}$. Since we choose a_1 as an odd permutation (see Table 3, again), we have $\mathrm{Mon}(f) = \mathfrak{S}_5$. As we will see in section 2, we can obtain generating vectors for smaller transitive groups of degree 5; nevertheless, as we just showed, the whole symmetric group can be generated in almost every case: only nonzero values for g_Y and n are required. Summarizing, we have the following corollary.

Corollary IV.2. *Consider a compact Riemann surface Y with $g_Y \geq 1$. Choose points $y_1, \dots, y_n$ in Y and a realizable tuple of ramification types $(t_1, \dots, t_n)$. If $n > 0$, then there is a holomorphic map $f : X \to Y$ with monodromy group $\mathfrak{S}_5$ and branch values $y_1, \dots, y_n$ of types $t_1, \dots, t_n$, respectively.*

The situation becomes a bit more complicated when $g_Y = 0$; we now have a restriction given by Riemann–Hurwitz formula: we have

$$2g_X - 2 = 5(2g_Y - 2) + \deg(R_f),$$

but $g_Y = 0$, so

$$2g_X = -8 + \deg(R_f),$$

and, since $g_X \geq 0$, we must have

$$8 \leq \deg(R_f). \tag{IV.2}$$

Recall that $\deg(R_f)$ can be directly computed from $(t_1, \dots, t_n)$; hence, if $t_i = [\nu_{i,1}, \dots, \nu_{i,k_i}]$ for each $i \in \{1, \dots, n\}$, then we define $\deg(t_1, \dots, t_n) := \sum_{i=1}^{n} \sum_{j=1}^{k_i} (\nu_{i,j} - 1)$ and call $\deg(t_1, \dots, t_n)$ the *degree* of the tuple $(t_1, \dots, t_n)$. Motivated by the necessary inequality (IV.2), we say that $(t_1, \dots, t_n)$ satisfies the *R–H condition* if $\deg(t_1, \dots, t_n) \geq 8$. The following theorem needs some technical lemmata that will also be useful in section 2.

Theorem IV.3. *For an arbitrary set $\{y_1, \dots, y_n\}$ of points in $\mathbb{P}^1$, there is a holomorphic map $f : X \to Y$ of degree 5 with branch values $y_1, \dots, y_n$ of types $t_1, \dots, t_n$, respectively, if and only if the tuple $(t_1, \dots, t_n)$ is even and satisfies the R–H condition.*

Lemma IV.4. *Consider an even tuple of cycle structures $(t_1, \dots, t_n)$ of permutations in $\mathfrak{S}_5$ with $n \geq 2$ and at least one even permutation. For any even cycle structure t, there are permutations $c_1, \dots, c_n$ such that each c_i is of type t_i and $\prod_{i=1}^{n} c_i$ is of type t.*

Proof. Without loss of generality, we may assume that t_1 is one of the even permutations in $(t_1, \ldots, t_n)$ (by hypothesis there is at least one). For each $i \in \{2, \ldots, n\}$, chose a permutation c_i of type t_i. Since the tuple $(t_1, \ldots, t_n)$ is even, the permutation $\prod_{i=2}^n c_i$ is also even; thus, modulo conjugation in $\mathfrak{S}_5$, we can assume that $\prod_{i=2}^n c_i$ is $(1\,2\,3\,4\,5)$, $(1\,2)(3\,4)$ or $(1\,2\,3)$.

Choose c_1 depending to the cycle structures t and t_1 and the value of $\prod_{i=2}^n c_i$ as in Table 4. Thereby, the permutation $\prod_{i=1}^n c_i$ is of type t and each c_i of type t_i. □

Lemma IV.5. *Consider an even tuple of cycle structures* $(t_1, \ldots, t_n)$ *of permutations in* $\mathfrak{S}_5$ *such that* $\deg(t_1, \ldots, t_n) \geq 4$. *There are permutations* $c_1, \ldots, c_n$ *such that each* c_i *is of type* t_i *and* $\prod_{i=1}^n c_i$ *is of type* $[5]$.

Proof. Suppose that there is at least one even permutation in $(t_1, \ldots, t_n)$, say t_1. Since $\deg(t_1, \ldots, t_n) \geq 4$, if $n = 1$, then $t_1 = [5]$ and the lemma is trivially satisfied. If $n \geq 2$, then Theorem IV.4 implies the existence of the c_i permutations. That proves the lemma restricted to the case where there are even cycle structures in $(t_1, \ldots, t_n)$.

Now suppose that there are no even permutations in $(t_1, \ldots, t_n)$, so $n \geq 2$. Suppose that at least one t_i, namely t_1, is of type $[4, 1]$ or $[3, 2]$. Set c_i as any permutation with cycle structure t_i for $i \in \{2, \ldots, n\}$. Since $(t_1, \ldots, t_n)$ is even, the product $\prod_{i=2}^n c_i$ must be an odd permutation; hence, its cycle structure is $[2, 1, 1, 1]$, $[4, 1]$ or $[3, 2]$. Thereby, up to conjugacy in $\mathfrak{S}_5$, we can assume that $\prod_{i=2}^n c_i$ is $(1\,2)$, $(1\,2\,3\,4)$ or $(1\,2\,3)(4\,5)$; for each case, choose t_1 according to its type as in Table 5. In this manner, the product $\prod_{i=1}^n c_i$ is of type $[5]$ and each c_i has cycle structure t_i. That proves the lemma restricted to the case where there cycle structures $[4, 1]$ or $[3, 2]$ in $(t_1, \ldots, t_n)$.

Finally, suppose that $t_i = [2, 1, 1, 1]$ for each $i \in \{1, \ldots, n\}$ (there are neither even nor type $[4, 1]$ or $[3, 2]$ permutations in $(t_1, \ldots, t_n)$). Since $\deg(t_1, \ldots, t_n) \geq 4$, we have that $n \geq 4$. Set $c_1 = (3\,4)$, $c_2 = (2\,3)$ and $c_3 = (1\,2)$; thus $\prod_{i=1}^3 c_i = (1\,2\,3\,4)$. Applying the already proved restricted version of the lemma to the tuple $([4, 1], t_4, \ldots, t_n)$ yields permutations $c_4, \ldots, c_n$ of type $[2, 1, 1, 1]$ such that $\prod_{i=1}^n c_i$ is of type $[5]$. □

Remark IV.1. Conjugating in $\mathfrak{S}_5$, we can choose the c_i permutations of Theorem IV.5 such that $\prod_{i=1}^n c_i$ is any prescribed permutation with cycle structure $[5]$.

Proof of Theorem IV.3. The sufficiency of the hypothesis is direct: the R–H condition is given by inequality (IV.2), and $(t_1, \ldots, t_n)$ is proved to be even in the same manner as in the proof of Theorem IV.1.

For proving the necessity of the hypothesis, we give explicit constructions in the several possible cases. Fix points $y_1, \ldots, y_n$ in $\mathbb{P}^1$ and ramification types $t_1, \ldots, t_n$ for those points; assume that $(t_1, \ldots, t_n)$ is even and satisfies the R–H condition. According to Theorem I.13, giving a holomorphic map $f\colon X \to \mathbb{P}^1$ of degree 5 with branch values $y_1, \ldots, y_n$ of types $t_1, \ldots, t_n$, respectively, is equivalent to give a generating $(0; m_1, \ldots, m_n)$-vector $(c_1, \ldots, c_n)$ of a transitive subgroup, namely $\mathrm{Mon}(f)$, of $\mathfrak{S}_5$ such that c_i has cycle structure t_i for each $i \in \{1, \ldots, n\}$.

TABLE 4. Product of permutations with prescribed cycle structure

t	$\prod_{i=2}^{n} c_i$	t_1	c_1	$\prod_{i=1}^{n} c_i$
[5]	(1 2 3 4 5)	[5]	(1 2 3 4 5)	(1 3 5 2 4)
		[2, 2, 1]	(1 3)(2 4)	(1 4 5 3 2)
		[3, 1, 1]	(1 2 5)	(1 5 2 3 4)
	(1 2)(3 4)	[5]	(1 3 2 4 5)	(1 4 2 3 5)
		[2, 2, 1]	(1 5)(2 3)	(1 3 4 2 5)
		[3, 1, 1]	(1 5 3)	(1 2 5 3 4)
	(1 2 3)	[5]	(1 2 5 4 3)	(1 5 4 3 2)
		[2, 2, 1]	(1 5)(3 4)	(1 2 4 3 5)
		[3, 1, 1]	(2 4 5)	(1 4 5 2 3)
[2,2,1]	(1 2 3 4 5)	[5]	(1 3 2 4 5)	(1 4)(3 5)
		[2, 2, 1]	(1 5)(2 4)	(1 4)(2 3)
		[3, 1, 1]	(1 4 3)	(1 2)(4 5)
	(1 2)(3 4)	[5]	(1 4 5 3 2)	(2 4)(3 5)
		[2, 2, 1]	(1 4)(2 3)	(1 3)(2 4)
		[3, 1, 1]	(1 2 5)	(1 5)(3 4)
	(1 2 3)	[5]	(1 4 3 2 5)	(1 5)(3 4)
		[2, 2, 1]	(2 3)(4 5)	(1 3)(4 5)
		[3, 1, 1]	(1 5 3)	(1 2)(3 5)
[3,1,1]	(1 2 3 4 5)	[5]	(1 4 5 3 2)	(3 5 4)
		[2, 2, 1]	(1 5)(3 4)	(1 2 4)
		[3, 1, 1]	(1 5 4)	(1 2 3)
	(1 2)(3 4)	[5]	(1 5 4 3 2)	(2 5 4)
		[2, 2, 1]	(1 5)(3 4)	(1 2 5)
		[3, 1, 1]	(1 3 4)	(1 2 3)
	(1 2 3)	[5]	(1 5 4 3 2)	(3 5 4)
		[2, 2, 1]	(1 5)(2 3)	(1 3 5)
		[3, 1, 1]	(1 2 3)	(1 3 2)

In any of the following cases, probably after a re-enumeration, we can take an even sub-tuple $(t_1, \ldots, t_k)$ of $(t_1, \ldots, t_n)$ with degree 4:

- If there is a cycle structure of degree 4, namely $[5]$, in $(t_1, \ldots, t_n)$.
- If there are at least two cycle structures of degree 2, namely $[3, 1, 1]$ or $[2, 2, 1]$.
- If there are at least one cycle structure of degree 3, namely $[4, 1]$ or $[3, 2]$, and one of degree 1, namely $[2, 1, 1, 1]$.

Table 5. Product of odd permutations with cycle structure [5]

$\prod_{i=2}^{n} c_i$	t_1	c_1	$\prod_{i=1}^{n} c_i$
(1 2)	[3, 2]	(1 5 3)(2 4)	(1 4 2 5 3)
	[4, 1]	(1 5 3 4)	(1 2 5 3 4)
(1 2 3 4)	[3, 2]	(1 3 4)(2 5)	(1 5 2 4 3)
	[4, 1]	(2 3 4 5)	(1 3 5 2 4)
(1 2 3)(4 5)	[3, 2]	(1 5 3)(2 4)	(1 4 3 5 2)
	[4, 1]	(1 5 2 3)	(1 3 5 4 2)

- If there are at least four cycle structures of degree 1.

In any of these cases, by Theorem IV.5, we can choose permutations $c_1, \dots, c_k$ of types $t_1, \dots, t_k$, respectively, such that $\prod_{i=1}^{k} c_i = (1\,2\,3\,4\,5)$. Since $(t_1, \dots, t_n)$ satisfies the R–H condition, we have that $\deg(t_{k+1}, \dots, t_n) = \deg(t_1, \dots, t_n) - \deg(t_1, \dots, t_k) \geq 4$; hence, again by Theorem IV.5, we can choose $c_{k+1}, \dots, c_n$ with cycle structures $t_{k+1}, \dots, t_n$, respectively, such that $\prod_{i=k+1}^{n} c_i = (1\,5\,4\,3\,2)$. Since $(1\,2\,3\,4\,5) \in \langle c_1, \dots, c_n \rangle$, the tuple $(c_1, \dots, c_n)$ is a generating $(0; m_1, \dots, m_n)$-vector of a transitive subgroup of $\mathfrak{S}_5$.

If $(t_1, \dots, t_n)$ does not satisfy any of the above conditions, then every t_i must have degree 3 except maybe for one cycle structure of degree 2, say t_n. The R–H conditions implies that $n \geq 3$. If $n \geq 4$, Theorem IV.5 can be applied separately to the sub-tuples (t_1, t_2) and $(t_3, \dots, t_n)$ yielding permutations $c_1, \dots, c_n$ of types $t_1, \dots, t_n$, respectively, such that $c_1 c_2 = (1\,2\,3\,4\,5)$ and $\prod_{i=3}^{n} c_i = (1\,5\,4\,3\,2)$. Since $(1\,2\,3\,4\,5) \in \langle c_1, \dots, c_n \rangle$, the tuple $(c_1, \dots, c_n)$ is a generating $(0; m_1, \dots, m_n)$-vector of a transitive subgroup of $\mathfrak{S}_5$. Now we deal with the special case where $n = 3$ (so $\deg(t_1) = \deg(t_2) = 3$ and $\deg(t_3) = 2$). If there is at least one $[3, 2]$ cycle structure in (t_1, t_2, t_3) set it as t_1. Choose c_1 to be $(1\,2\,3)(4\,5)$ or $(1\,2\,3\,4)$ according to t_1 and then choose c_2 according to t_3 and c_1 as given by Table 6; set $c_3 := c_2^{-1} c_1^{-1}$. Thereby, in each sub-case, the permutation c_i has cycle structure t_i for each $i \in \{1, 2, 3\}$ and $c_1 c_2 c_3$ is the identity. Moreover, column [5] of Table 6 gives a permutation of type [5] that belongs to $\langle c_1, c_2, c_3 \rangle$ for each of the six sub-cases; therefore, the tuple (c_1, c_2, c_3) is a generating $(0; m_1, m_2, m_3)$-vector of a transitive subgroup of $\mathfrak{S}_5$. □

2. Monodromy group in terms of the ramification data

In this section, we list all possible monodromy groups $\mathrm{Mon}(f)$ modulo conjugacy in $\mathfrak{S}_5$ for the degree 5 covering $f\colon X \to Y$ and give criteria to determine $\mathrm{Mon}(f)$ in terms of the ramification data of f; as in section 1, these criteria will depend on g_Y.

Proposition IV.6. *Let $f\colon X \to Y$ be a degree 5 covering map between compact Riemann surfaces. The monodromy group* $\mathrm{Mon}(f)$ *is conjugate to one of the following subgroups of* $\mathfrak{S}_5$:

(1) *The cyclic group* $\langle (1\ 2\ 3\ 4\ 5) \rangle$*, denoted by* C_5*, of order* 5.

TABLE 6. Odd permutations with product of degree 2 cycle structure

t_3	c_1	t_2	c_2	c_1c_2	[5]
[2,2,1]	(1 2 3)(4 5)	[3, 2]	(1 4 2)(3 5)	(1 5)(3 4)	$c_2^{-1}c_1c_2^{-2}c_1^2$
		[4, 1]	(2 3 4 5)	(1 2)(3 5)	$c_1^{-2}c_2^{-1}c_1^3c_2c_1^{-1}c_1^{-2}$
	(1 2 3 4)	[4, 1]	(2 4 5 3)	(1 2)(4 5)	$c_1c_2^{-1}$
[3,1,1]	(1 2 3)(4 5)	[3, 2]	(1 4 5)(2 3)	(1 5 2)	$c_2c_1^2c_2^3$
		[4, 1]	(2 5 4 3)	(1 2 4)	$c_1^{-1}c_2^{-1}c_1^{-2}c_2^{-1}c_1^3$
	(1 2 3 4)	[4, 1]	(2 5 4 3)	(1 2 5)	$c_1^{-1}c_2c_1^{-1}c_2$

(2) *The dihedral group* $\langle(1\ 2\ 3\ 4\ 5), (2\ 5)(3\ 4)\rangle$*, denoted by* D_5*, of order* 10.
(3) *The group* $\langle(1\ 2\ 3\ 4\ 5), (2\ 3\ 5\ 4)\rangle$*, isomorphic to the general affine group* $\mathrm{Aff}(\mathbb{F}_5)$ *of a* 1*-dimensional affine space over* $\mathbb{F}_5$*, of order* 20*. It will be denoted just by* $\mathrm{Aff}(\mathbb{F}_5)$.
(4) *The alternating group* $\langle(1\ 2\ 3\ 4\ 5), (1\ 3\ 4\ 5\ 2)\rangle$*, denoted by* $\mathfrak{A}_5$*, of order* 60.
(5) *The symmetric group* $\langle(1\ 2\ 3\ 4\ 5), (1\ 2)\rangle$*, denoted by* $\mathfrak{S}_5$*, of order* 120.

PROOF. According to Theorem I.8, the monodromy group of f is a transitive subgroup of $\mathfrak{S}_5$; those subgroups and their generators are tabulated in [**5**, Tables 5A and 5B]. □

The cycle structure of the elements of each group listed in Theorem IV.6 are tabulated in [**5**, Table 5C]; Table 7 summarizes those cycle structures. The results in [**5**] are implemented in the computer algebra system SageMath through the GAP package TransGrp (see [**11**, **12**, **26**]).

Corollary IV.7. *Each transitive subgroup of* $\mathfrak{S}_5$ *can be generated by two elements.*

REMARK IV.2. The isomorphism of item (3) of Theorem IV.6 is, indeed, very natural: each permutation $\sigma \in \langle(1\ 2\ 3\ 4\ 5), (2\ 3\ 5\ 4)\rangle$ corresponds to the affine map that, for each $n \in \{1, 2, 3, 4, 5\}$, maps $[n]_5$ to $[\sigma(n)]_5$; in particular, the two generators $(1\,2\,3\,4\,5)$ and $(2\,3\,5\,4)$ corresponds to the maps $x \mapsto x + 1$ and $x \mapsto 2x + 4$, respectively.

In order to give a more straightforward proof of Theorem IV.10, we now establish some auxiliary lemmata.

Lemma IV.8. *For a nonempty finite subset* $\{c_1, \ldots, c_n\}$ *of permutations in* D_5 *with exactly* m *permutations of cycle structure* $[2, 2, 1]$*, the product* $\prod_{i=1}^n c_i$ *is:*
(1) *of type* $[2, 2, 1]$ *if* m *is odd*; *or*
(2) *of type* $[5]$ *or the identity if* m *is even.*

PROOF. Consider the natural isomorphism $\phi\colon \mathrm{D}_5 \to \mathrm{D}_5/\mathrm{C}_5$ given by quotient. The image by ϕ of any type $[5]$ permutation is the identity, while the image of a type $[2, 2, 1]$ permutation is the only non-trivial element $(1\,2)(3\,4)\mathrm{C}_5$ of $\mathrm{D}_5/\mathrm{C}_5$. We have $\phi(\prod_{i=1}^n c_i) =$

TABLE 7. Cycle structure of permutations in each possible monodromy group of a holomorphic map f of degree 5

$\mathrm{Mon}(f)$ up to isomorphism	Cycle structure of permutations in $\mathrm{Mon}(f)$
Cyclic group C_5	$[5]$
Dihedral group D_5	$[5]$ or $[2,2,1]$
Affine group $\mathrm{Aff}(\mathbb{F}_5)$	$[5]$, $[4,1]$ or $[2,2,1]$
Alternating group $\mathfrak{A}_5$	$[5]$, $[3,1,1]$ or $[2,2,1]$
Symmetric group $\mathfrak{S}_5$	$[5]$, $[4,1]$, $[3,2]$, $[3,1,1]$, $[2,2,1]$ or $[2,1,1,1]$

$\prod_{i=1}^n \phi(c_i) = ((1\,2)(3\,4))^m \mathrm{C}_5$; therefore, whether or not $\prod_{i=1}^n c_i$ belongs to C_5 depends solely on the parity of m. □

Lemma IV.9. *For a nonempty tuple $(t_1, \ldots, t_n)$ of cycle structures $[2,2,1]$ and $[5]$ with an even number of cycle structures $[2,2,1]$, we can choose a permutation c_i in D_5 of type t_i for each $i \in \{1, \ldots, n\}$ such that $\prod_{i=1}^n c_i$ is of type $[5]$.*

PROOF. We proceed by induction. If $n = 1$, set $c_1 := (1\,2\,3\,4\,5)$. If $n = 2$, we have two cases: If there are permutations of type $[2,2,1]$, then $t_1 = t_2 = [2,2,1]$, so we set $c_1 := (2\,5)(3\,4)$ and $c_2 := (1\,5)(2\,4)$, and hence $c_1 c_2 = (1\,2\,3\,4\,5)$. If there are no permutations of type $[2,2,1]$, then $t_1 = t_2 = [5]$, so we set $c_1 := (1\,2\,3\,4\,5)$ and $c_2 := (1\,2\,3\,4\,5)$, and hence $c_1 c_2 = (1\,3\,5\,2\,4)$.

Now consider $n \geq 3$. If there is at least one type $[5]$ in $(t_1, \ldots, t_n)$, say t_n, we can choose, by inductive hypothesis, permutations c_i of type t_i for $i \in \{1, \ldots, n-1\}$ such that $\prod_{i=1}^{n-1} c_i$ is of type $[5]$, say $\prod_{i=1}^{n-1} c_i = (1\,2\,3\,4\,5)$. Set $c_n := (1\,2\,3\,4\,5)$, so $\prod_{i=1}^n c_i = (1\,3\,5\,2\,4)$. If there are no permutations of type $[5]$, we can choose, by inductive hypothesis, permutations c_i of type t_i for $i \in \{1, \ldots, n-2\}$ such that $\prod_{i=1}^{n-2} c_i$ is of type $[5]$, say $\prod_{i=1}^{n-2} c_i = (1\,2\,3\,4\,5)$. So we set $c_{n-1} := (2\,5)(3\,4)$ and $c_n := (1\,5)(2\,4)$, and hence $\prod_{i=1}^n c_i = (1\,3\,5\,2\,4)$. □

Theorem IV.10. *Consider a degree 5 holomorphic map $f\colon X \to Y$ between compact Riemann surfaces with ramification data $(t_1, \ldots, t_n)$. If $g_Y \geq 1$, then the following statements hold:*

(1) *Suppose that $n = 0$. If $g_Y = 1$, then $\mathrm{Mon}(f) \cong \mathrm{C}_5$. If $g_Y > 1$, then $\mathrm{Mon}(f)$ may be any transitive subgroup of $\mathfrak{S}_5$.*
(2) *If at least one t_i equals $[3,2]$ or $[2,1,1,1]$, then $\mathrm{Mon}(f) \cong \mathfrak{S}_5$.*
(3) *If there are types $[3,1,1]$ and $[4,1]$ in $(t_1, \ldots, t_n)$, then $\mathrm{Mon}(f) \cong \mathfrak{S}_5$.*
(4) *If there are no types $[3,2]$, $[3,1,1]$ or $[2,1,1,1]$ in $(t_1, \ldots, t_n)$, but there is at least one type $[4,1]$, then $\mathrm{Mon}(f) \cong \mathfrak{S}_5$ or $\mathrm{Mon}(f) \cong \mathrm{Aff}(\mathbb{F}_5)$.*
(5) *If f has only even branch values and at least one of them is of type $[3,1,1]$, then $\mathrm{Mon}(f) \cong \mathfrak{S}_5$ or $\mathrm{Mon}(f) \cong \mathfrak{A}_5$.*
(6) *Suppose that there are neither odd nor $[3,1,1]$ types in $(t_1, \ldots, t_n)$, but there is at least one type $[2,2,1]$. There are three cases:*

(a) *If* $n = 1$, $t_1 = [2, 2, 1]$ *and* $g_Y = 1$; *then* $\mathrm{Mon}(f) \cong \mathfrak{S}_5$.
(b) *If there is an odd number of types* $[2, 2, 1]$ *in* $(t_1, \ldots, t_n)$, *and* $n > 1$ *or* $g_Y > 1$; *then* $\mathrm{Mon}(f) \cong \mathfrak{S}_5$ *or* $\mathrm{Mon}(f) \cong \mathfrak{A}_5$.
(c) *If there is an even number of types* $[2, 2, 1]$ *in* $(t_1, \ldots, t_n)$, *then* $\mathrm{Mon}(f)$ *is conjugate to any transitive subgroup of* $\mathfrak{S}_5$ *but* C_5.

(7) *Suppose that* $t_i = [5]$ *for each* $i \in \{1, \ldots, n\}$. *There are two cases*:
(a) *If* $n = 1$, *then* $\mathrm{Mon}(f)$ *is conjugate to any transitive subgroup of* $\mathfrak{S}_5$ *but* C_5.
(b) *If* $n > 1$, *then* $\mathrm{Mon}(f)$ *is any transitive subgroup of* $\mathfrak{S}_5$.

*Conversely, for every realizable tuple of ramification types given by Theorem IV.*1 *and for each possible monodromy group previously stated, there is a holomorphic map of degree* 5 *with those ramification data and monodromy group.*

Proof. According to Theorem I.13, the holomorphic map $f\colon X \to Y$ with ramification data $(t_1, \ldots, t_n)$ exists if and only if the group $\mathrm{Mon}(f)$ has a generating $(g_Y; m_1, \ldots, m_n)$-vector $(a_1, \ldots, a_{g_Y}, b_1, \ldots, b_{g_Y}, c_1, \ldots, c_n)$ such that c_i has cycle structure t_i for each $i \in \{1, \ldots, n\}$. In particular, each c_i belongs to $\mathrm{Mon}(f)$; therefore, $\mathrm{Mon}(f)$ contains permutations of each type t_i. We will separately prove items (1) to (7) by assuming the existence of such a generating vector. After the proof of each item, we show the existence of a map f with the stated ramification data and monodromy group; usually, by explicitly constructing a generating vector with suitable properties. In the cases where $\mathrm{Mon}(f) = \mathfrak{S}_5$, existence of f is given by Theorem IV.2; hence, in those cases the existence proof is omitted.

Item (1). If $g_Y = 1$, then $\pi_1(Y, y)$ is abelian; hence, the monodromy group $\mathrm{Mon}(f)$ (the image of $\pi_1(Y, y)$ by the monodromy representation) is also abelian. Therefore, by Theorem IV.6, it must be conjugate to C_5. For the existence proof of a suitable map, note that, since we just proved that there is only one possible monodromy group for a holomorphic map with the prescribed characteristics, namely C_5, the map f given by Theorem IV.1, which in this case is unramified, must satisfy $\mathrm{Mon}(f) \cong \mathrm{C}_5$.

Item (2). Table 7 shows that the only transitive subgroup of $\mathfrak{S}_5$ that contains permutations of type $[3, 2]$ or $[2, 1, 1, 1]$ is $\mathfrak{S}_5$ itself; hence $\mathrm{Mon}(f) \cong \mathfrak{S}_5$.

Item (3). The only group in Table 7 that contains permutations with both cycle structures, $[3, 1, 1]$ and $[4, 1]$, is $\mathfrak{S}_5$; hence $\mathrm{Mon}(f) \cong \mathfrak{S}_5$.

Item (4). As in the two above items, according to Table 7, we have $\mathrm{Mon}(f) = \mathfrak{S}_5$ or $\mathrm{Mon}(f) \cong \mathrm{Aff}(\mathbb{F}_5)$. For existence in the case where $\mathrm{Mon}(f) \cong \mathrm{Aff}(\mathbb{F}_5)$, we can assume, modulo conjugation, that $\mathrm{Mon}(f) = \mathrm{Aff}(\mathbb{F}_5)$. Set c_i as an arbitrary permutation of type t_i in $\mathrm{Aff}(\mathbb{F}_5)$ for each $i \in \{3, \ldots, n\}$; since $\mathrm{Aff}(\mathbb{F}_5) \cap \mathfrak{A}_5 = \mathrm{D}_5$, we have that $\prod_{i=3}^{n} c_i$ is equal to Id, $(1\,4)(2\,3)$ or $(1\,2\,3\,4\,5)$ up to conjugacy. Set c_1 and c_2 as the permutations corresponding to the affine maps $x \mapsto 2x$ and $x \mapsto 3x$; $x \mapsto 2x$ and $x \mapsto 2x$; or $x \mapsto 3x + 3$ and $2x \mapsto 2x$ according to each possible $\prod_{i=3}^{n} c_i$ mentioned above. In this manner $\prod_{i=1}^{n} c_i = \mathrm{Id}$ and $\mathrm{Aff}(\mathbb{F}_5)$ has the generating $(g_Y; m_1, \ldots, m_n)$-vector $((1\,2\,3\,4\,5), \underbrace{\mathrm{Id}, \ldots, \mathrm{Id}}_{2g-1}, c_1, \ldots, c_n)$.

Item (5). According to Table 7, we have $\mathrm{Mon}(f) = \mathfrak{S}_5$ or $\mathrm{Mon}(f) = \mathfrak{A}_5$.

Now we prove existence in the case where $\mathrm{Mon}(f) = \mathfrak{A}_5$. By Theorem IV.4, if $n \geq 2$ we can choose a permutation $c_i \in \mathfrak{A}_5$ of type t_i for each $i \in \{1, \ldots, n\}$ such that $\prod_{i=1}^{n} c_i$ is of type $[3, 1, 1]$; since there is at least one permutation of type $[3, 1, 1]$, in case $n = 1$, we still can chose (the only) c_i in the above manner. Without loss of generality, suppose $\prod_{i=1}^{n} c_i = (1\,3\,2)$. Since $\langle (1\,5\,2\,3\,4), (1\,3\,4\,2\,5)\rangle = \mathfrak{A}_5$ (they are both even and generate a transitive subgroup of order 25 or greater, see Theorem IV.6) and $[(1\,5\,2\,3\,4), (1\,3\,4\,2\,5)] = (1\,2\,3)$, the group $\mathfrak{A}_5$ has the suitable generating $(g_Y; m_1, \ldots, m_n)$-vector

$$((1\,5\,2\,3\,4), \underbrace{\mathrm{Id}, \ldots, \mathrm{Id}}_{g_Y-1}, (1\,3\,4\,2\,5), \underbrace{\mathrm{Id}, \ldots, \mathrm{Id}}_{g_Y-1}, c_1, \ldots, c_n).$$

Item (6). According to Table 7, the monodromy group $\mathrm{Mon}(f)$ could be isomorphic to any transitive subgroup of $\mathfrak{S}_5$ but C_5, which implies item (6c). Suppose there is an odd number of branch values of type $[2, 2, 1]$ and $\mathrm{Mon}(f)$ is D_5 or $\mathrm{Aff}(\mathbb{F}_5)$ modulo conjugacy; let $(a_1, \ldots, a_{g_Y}, b_1, \ldots, b_{g_Y}, c_1, \ldots, c_n)$ be a generating vector of $\mathrm{Mon}(f)$; thereby, the permutation c_i is of type $[5]$ or $[2, 2, 1]$ for each $i \in \{1, \ldots, n\}$. Note that D_5 is a subgroup of $\mathrm{Aff}(\mathbb{F}_5)$ and that every permutation of type $[2, 2, 1]$ or $[5]$ in $\mathrm{Aff}(\mathbb{F}_5)$ actually belongs to D_5; hence, we have $c_i \in \mathrm{D}_5$ for each $i \in \{1, \ldots, n\}$. Theorem IV.8 implies that $\prod_{i=1}^{n} c_i$ is of type $[2, 2, 1]$. On the other hand, we have $\mathrm{Aff}(\mathbb{F}_5)' = \mathrm{D}_5' = \mathrm{C}_5$; therefore, by equation (IV.1), there is a permutation of type $[2, 2, 1]$ in C_5, which contradicts Table 7. That proves item (6b). If $n = 1$, $t_1 = [2, 2, 1]$ and $g_Y = 1$, then Theorem I.11 implies that $g_{\hat{X}} = 16$; however, according to the classification of group actions on surfaces of low genus (up to 48) in [**3**, chapter 5] implemented in GAP [**11**], there is no action of $\mathfrak{A}_5$ with signature $(1; 2)$ on a Riemann surface of genus 16.

Existence for a map f as in item (6b) with $\mathrm{Mon}(f) = \mathfrak{A}_5$ where $g_Y > 1$ is granted by the following generating $(g_Y; 2)$-vector of $\mathfrak{A}_5$:

$$((1\,3\,4\,2\,5), (1\,3\,2\,4\,5), \underbrace{\mathrm{Id}, \ldots, \mathrm{Id}}_{g_Y-2}, (1\,5\,2\,3\,4), (1\,5\,4\,3\,2), \underbrace{\mathrm{Id}, \ldots, \mathrm{Id}}_{g_Y-2}, (1\,2)(3\,4)).$$

If $g_Y = 1$ and $n > 1$, then, according to Theorem IV.4, we can choose a permutation c_i of type t_i for each $i \in \{1, \ldots, n\}$ such that $\prod_{i=1}^{n} c_i = (1\,3\,2)$; thereby, we get the following generating $(1; m_1, \ldots, m_n)$-vector of $\mathfrak{A}_5$:

$$((1\,4\,3\,2\,5), (1\,5\,2\,4\,3), c_1, \ldots, c_n).$$

For existence in the case of item (6c), by Theorem IV.9, we can set c_i as a permutation of type t_i in D_5 for each $i \in \{1, \ldots, n\}$ such that $\prod_{i=1}^{n} c_i = (1\,5\,4\,3\,2)$; note that $\mathrm{D}_5 \subset \mathrm{Aff}(\mathbb{F}_5)$ and $\mathrm{D}_5 \subset \mathfrak{A}_5$. Set a_1 and b_1 according to $\mathrm{Mon}(f)$ as given by Table 8, also set $a_i := \mathrm{Id}$ and $b_i := \mathrm{Id}$ for $i \in \{2, \ldots, g_Y\}$; so $\prod_{i=1}^{g_Y} [a_i, b_i] = [a_1, b_1] = (1\,2\,3\,4\,5)$ and $\langle a_1, b_1\rangle = \mathrm{Mon}(f)$ in each case. Thereby, we have a suitable generating $(g_Y; m_1, \ldots, m_n)$-vector for $\mathrm{Mon}(f)$.

Item (7). Suppose that f has only one branch value, of type $[5]$, and $\mathrm{Mon}(f) = \mathrm{C}_5$. Equation (IV.1) states that there is a permutation with cycle structure $[5]$ in $\mathrm{C}_5' = \{\mathrm{Id}\}$, a contradiction. That contradiction implies item (7a).

TABLE 8. Permutations with commutator of type [5]

$\mathrm{Mon}(f)$	a_1	b_1	$[a_1, b_1]$
$\mathfrak{A}_5$	$(1\,2\,5\,3\,4)$	$(1\,5\,3\,2\,4)$	$(1\,2\,3\,4\,5)$
$\mathrm{Aff}(\mathbb{F}_5)$	$(1\,3\,2\,5)$	$(3\,2\,4\,5)$	$(1\,2\,3\,4\,5)$
D_5	$(1\,3\,5\,2\,4)$	$(1\,2)(3\,5)$	$(1\,2\,3\,4\,5)$

For existence in the case where $\mathrm{Mon}(f) \cong \mathrm{C}_5$, according to Theorem IV.9, we can choose a permutation c_i of type [5] in C_5 for each $i \in \{2, \ldots, n\}$ such that $\prod_{i=2}^{n} c_i = (1\,2\,3\,4\,5)$. Set $c_1 := (1\,5\,4\,3\,2)$ and each a_i and b_i as Id; thereby, we get a generating $(g_Y; 5, \ldots, 5)$-vector of C_5. When $\mathrm{Mon}(f)$ is D_5, $\mathrm{Aff}(\mathbb{F}_5)$ or $\mathfrak{A}_5$, according to Theorem IV.9 again, we can choose a permutation c_i of type [5] in C_5 for each $i \in \{2, \ldots, n\}$ such that $\prod_{i=2}^{n} c_i = (1\,5\,4\,3\,2)$. Set a_1 and b_1 according to $\mathrm{Mon}(f)$ as given by Table 8, also set $a_i := \mathrm{Id}$ and $b_i := \mathrm{Id}$ for $i \in \{2, \ldots, g_Y\}$; so $\prod_{i=1}^{g_Y} [a_i, b_i] = (1\,2\,3\,4\,5)$ and $\langle a_1, b_1 \rangle = \mathrm{Mon}(f)$ in each case. Thereby, we have a suitable generating $(g_Y; 5, \ldots, 5)$-vector for $\mathrm{Mon}(f)$. □

Theorem IV.10 implies directly the following result, which is just a restatement of the converses of items (1) to (7).

Corollary IV.11. *Consider a degree 5 holomorphic map $f : X \to Y$ between compact Riemann surfaces. If $g_Y \geq 1$, then the following statements hold*:

(1) *If* $\mathrm{Mon}(f) \cong \mathrm{C}_5$, *then*
$$R_f = \sum_{j=1}^{n_1} 4p_j,$$
where $n_1 > 1$ and $p_1, \ldots, p_n$ are different points in X.

(2) *If* $\mathrm{Mon}(f) \cong \mathrm{D}_5$, *then*
$$R_f = \sum_{j=1}^{n_1} 4p_j + \sum_{j=1}^{n_2} (q_j + r_j),$$
where n_2 is even and $p_1, \ldots, p_{n_1}, q_1, \ldots, q_{n_2}, r_1, \ldots, r_{n_2}$ are different points in X such that $f(q_j) = f(r_j)$ for each $j \in \{1, \ldots, n_2\}$. If $g_Y = 1$, then n_1 and n_2 cannot be both zero.

(3) *If* $\mathrm{Mon}(f) \cong \mathrm{Aff}(\mathbb{F}_5)$, *then*
$$R_f = \sum_{j=1}^{n_1} 4p_j + \sum_{j=1}^{n_2} (q_j + r_j) + \sum_{j=1}^{n_3} 3s_j,$$
where n_3 is even and $p_1, \ldots, p_{n_1}, q_1, \ldots, q_{n_2}, r_1, \ldots, r_{n_2}, s_1, \ldots, s_{n_3}$ are different points in X such that $f(q_j) = f(r_j)$ for each $j \in \{1, \ldots, n_2\}$. If $g_Y = 1$, then n_1, n_2, and n_3 cannot be all zero.

(4) *If* $\mathrm{Mon}(f) = \mathfrak{A}_5$, *then*

$$R_f = \sum_{j=1}^{n_1} 4p_j + \sum_{j=1}^{n_2}(q_j + r_j) + \sum_{j=1}^{n_4} 2t_j,$$

where $p_1, \dots, p_{n_1}, q_1, \dots, q_{n_2}, r_1, \dots, r_{n_2}, t_1, \dots, t_{n_4}$ *are different points in* X *such that* $f(q_j) = f(r_j)$ *for each* $j \in \{1, \dots, n_2\}$. *If* $g_Y = 1$, *then* n_1, n_2, *and* n_4 *cannot be all zero. Moreover, if* $g_Y = 1$ *and* $n_1 = n_4 = 0$, *then* $n_2 \geq 2$.

(5) *If* $\mathrm{Mon}(f) = \mathfrak{S}_5$, *then*

$$R_f = \sum_{j=1}^{n_1} 4p_j + \sum_{j=1}^{n_2}(q_j + r_j) + \sum_{j=1}^{n_3} 3s_j + \sum_{j=1}^{n_4} 2t_j + \sum_{j=1}^{n_5}(2u_j + v_j) + \sum_{j=1}^{n_6} w_j,$$

where n_3, n_5 *and* n_6 *are even and* $p_1, \dots, p_{n_1}$, $q_1, \dots, q_{n_2}$, $r_1, \dots, r_{n_2}$, $s_1, \dots, s_{n_3}$, $t_1, \dots, t_{n_4}$, $u_1, \dots, u_{n_5}$, $v_1, \dots, v_{n_5}$, $w_1, \dots, w_{n_6}$ *are different points in* X *such that* $f(q_j) = f(r_j)$ *for each* $j \in \{1, \dots, n_2\}$, $f(u_j) = f(v_j)$ *for each* $j \in \{1, \dots, n_5\}$, $f(t_j) \neq f(w_k)$ *for each pair* (j, k), *and* $f(w_j) \neq f(w_k)$ *for* $j \neq k$. *If* $g_Y = 1$, *then* n_i *cannot be zero for all* $i \in \{1, \dots, 6\}$.

Now we list possible monodromy groups $\mathrm{Mon}(f)$ according to the ramification of f in the special case where $g_Y = 0$; namely, when $Y \cong \mathbb{P}^1$. Recall that the ramification data of the map f must fulfill an additional condition when $g_Y = 0$ in order to be realizable; namely, the R–H condition (see Theorem IV.3).

Theorem IV.12. *Consider a degree 5 holomorphic map* $f \colon X \to \mathbb{P}^1$ *between compact Riemann surfaces with ramification data* $(t_1, \dots, t_n)$; *the following statements hold*:

(1) *If at least one* t_i *equals* $[2, 1, 1, 1]$ *or* $[3, 2]$, *then* $\mathrm{Mon}(f) = \mathfrak{S}_5$.
(2) *If there are types* $[3, 1, 1]$ *and* $[4, 1]$ *in* $(t_1, \dots, t_n)$, *then* $\mathrm{Mon}(f) = \mathfrak{S}_5$.
(3) *Suppose there are no types* $[3, 2]$, $[3, 1, 1]$ *or* $[2, 1, 1, 1]$ *in* $(t_1, \dots, t_n)$, *but there is at least one type* $[4, 1]$, *we have*:
  (*a*) *If* $\deg(t_1, \dots, t_n) = 8$, *then* $\mathrm{Mon}(f) \cong \mathrm{Aff}(\mathbb{F}_5)$.
  (*b*) *If* $\deg(t_1, \dots, t_n) > 8$, *then* $\mathrm{Mon}(f) \cong \mathrm{Aff}(\mathbb{F}_5)$ *or* $\mathrm{Mon}(f) = \mathfrak{S}_5$.
(4) *If* f *has only even branch values and at least one of them is of type* $[3, 1, 1]$, *then* $\mathrm{Mon}(f) = \mathfrak{A}_5$.
(5) *Suppose that there are neither odd nor* $[3, 1, 1]$ *types in* $(t_1, \dots, t_n)$, *but there is at least one type* $[2, 2, 1]$. *We have three cases*:
  (*a*) *If* $\deg(t_1, \dots, t_n) = 8$, *then* $\mathrm{Mon}(f) \cong \mathrm{D}_5$.
  (*b*) *If there is an odd number of branch values of type* $[2, 2, 1]$, *then* $\mathrm{Mon}(f) = \mathfrak{A}_5$.
  (*c*) *If* $\deg(t_1, \dots, t_n) > 8$ *and there is an even number of branch values of type* $[2, 2, 1]$, *then* $\mathrm{Mon}(f) \cong \mathrm{D}_5$ *or* $\mathrm{Mon}(f) = \mathfrak{A}_5$.
(6) *Suppose that* $t_i = [5]$ *for each* $i \in \{1, \dots, n\}$. *If* $n = 2$, *then* $\mathrm{Mon}(f) \cong \mathrm{C}_5$; *otherwise, we have* $\mathrm{Mon}(f) \cong \mathrm{C}_5$ *or* $\mathrm{Mon}(f) = \mathfrak{A}_5$.

Conversely, for any realizable tuple $(t_1, \dots, t_n)$ *of ramification types and for each possible monodromy group listed in items* (1) *to* (6) *according to* $(t_1, \dots, t_n)$*, there is a holomorphic map of degree* 5 *with that ramification data and that monodromy group.*

Proof. As in the proof of Theorem IV.10, according to Theorem I.13, the holomorphic map $f\colon X \to Y$ with ramification data $(t_1, \dots, t_n)$ exists if and only if the group $\mathrm{Mon}(f)$ has a generating $(0; m_1, \dots, m_n)$-vector $(c_1, \dots, c_n)$ such that c_i has cycle structure t_i for each $i \in \{1, \dots, n\}$. We will separately prove items (1) to (6) by assuming the existence of such a generating vector. After the proof of each item, we show the existence of a map f with the stated ramification data (assuming its realizability) and monodromy group. In the cases where there is only one possible monodromy group G up to conjugacy, the map f given by Theorem IV.3 already satisfies $\mathrm{Mon}(f) \cong G$; hence, in those cases, the existence proof is omitted.

Item (1). Table 7 shows that the only transitive subgroup of $\mathfrak{S}_5$ that contains permutations of type $[3, 2]$ or $[2, 1, 1, 1]$ is $\mathfrak{S}_5$ itself; hence $\mathrm{Mon}(f) = \mathfrak{S}_5$.

Item (2). The proof is identical to that of item (1).

Item (3). Table 7 shows that the only transitive subgroups of $\mathfrak{S}_5$ that contain permutations of type $[4, 1]$ are $\mathrm{Aff}(\mathbb{F}_5)$ and $\mathfrak{S}_5$; hence $\mathrm{Mon}(f) \cong \mathrm{Aff}(\mathbb{F}_5)$ or $\mathrm{Mon}(f) = \mathfrak{S}_5$, which proves item (3b). Now suppose that $\deg(t_1, \dots, t_n) = 8$, since $(t_1, \dots, t_n)$ is even, there must be at least two types $[4, 1]$, say t_1 and t_2, so $n = 3$ and $\deg(t_3) = 2$; hence Theorem I.11 yields $g_{\hat{X}} = 1$. Since the full group of automorphisms of a surface of genus 1 is isomorphic to $\mathbb{Z}^2 \rtimes \mathrm{C}_k$, where $k \in \{2, 4, 6\}$ (see, for example, [**3**, subsection 3.4]), which is a solvable group, the non-solvable group $\mathfrak{S}_5$ cannot act on $\hat{X}$. That proves item (3a).

Since now we have two possible monodromy groups, we give an existence proof for both cases separately.

Case $\mathrm{Mon}(f) = \mathfrak{S}_5$. Assume $t_1 = t_2 = [4, 1]$. Since $(t_3, \dots, t_n)$ is also even and $\deg(t_3, \dots, t_n) \geq 4$, according to Theorem IV.5, we can choose a permutation c_i of type t_i for each $i \in \{3, \dots, n\}$ such that $\prod_{i=3}^n c_i = (1\,5\,4\,3\,2)$. Set $c_1 := (1\,2\,5\,4)$ and $c_2 := (2\,5\,3\,4)$, then $\prod_{i=1}^n c_i = \mathrm{Id}$; also, since $c_2^{-1} c_1 = (1\,5\,4)$ and the only transitive subgroup of $\mathfrak{S}_5$ that contains $[3, 1, 1]$ and $[4, 1]$ element is $\mathfrak{S}_5$ itself, we have $\langle c_1, \dots, c_n \rangle = \mathfrak{S}_5$.

Case $\mathrm{Mon}(f) \cong \mathrm{Aff}(\mathbb{F}_5)$. Set c_i as any permutation of type t_i in $\mathrm{Aff}(\mathbb{F}_5)$ for $i \in \{3, \dots, n\}$. The product $\prod_{i=3}^n c_i$ must be of type $[2, 2, 1]$ or $[5]$ because it is even and an element of $\mathrm{Aff}(\mathbb{F}_5)$; without loss of generality, assume that it equals $(1\,4)(2\,3)$ or $(1\,5\,4\,3\,2)$ (in the notation of Remark IV.2, they correspond to the maps $x \mapsto -x$ and $x \mapsto x - 1$, respectively; in particular, they both belong to $\mathrm{Aff}(\mathbb{F}_5)$). Set

$$(c_1, c_2) := \begin{cases} ((1\,3\,2\,5), (1\,4\,5\,2)) & \text{if } \prod_{i=3}^n c_i = (1\,4)(2\,3), \\ ((2\,4\,5\,3), (1\,3\,2\,5)) & \text{if } \prod_{i=3}^n c_i = (1\,2\,3\,4\,5). \end{cases}$$

In the notation of Remark IV.2, those permutations correspond to the maps $x \mapsto 2x + 1$, $x \mapsto 2x + 2$, $x \mapsto 3x + 3$ and $x \mapsto 2x + 1$, respectively; in particular, they belong to $\mathrm{Aff}(\mathbb{F}_5)$. Since $\langle c_1, c_2 \rangle$ generates a transitive subgroup of $\mathfrak{S}_5$ contained in $\mathrm{Aff}(\mathbb{F}_5)$ and with type

$[4,1]$ permutations, we have $\langle c_1,\dots,c_n\rangle = \mathrm{Aff}(\mathbb{F}_5)$; so $(c_1,\dots,c_n)$ is a suitable generating $(0;4,4,t_3,\dots,t_n)$-vector of $\mathrm{Aff}(\mathbb{F}_5)$.

Item (4). According to Table 7, the monodromy group $\mathrm{Mon}(f)$ equals $\mathfrak{S}_5$ or $\mathfrak{A}_5$; but every even permutation in $\mathfrak{S}_5$ is contained in $\mathfrak{A}_5$, so there is no suitable generating vector for $\mathfrak{S}_5$. Therefore, $\mathrm{Mon}(f) = \mathfrak{A}_5$.

Item (5). According to Table 7, $\mathrm{Mon}(f)$ could be conjugate to any transitive subgroups of $\mathfrak{S}_5$ but C_5; however, neither $\mathfrak{S}_5$ nor $\mathrm{Aff}(\mathbb{F}_5)$ is generated by their even permutations because all of them are contained in $\mathfrak{A}_5$ and D_5, respectively. Hence $\mathrm{Mon}(f) = \mathfrak{A}_5$ or $\mathrm{Mon}(f) \cong D_5$.

Now suppose $\deg(t_1,\dots,t_n) = 8$, then, after a potential re-enumeration, there are two possibilities:

- $n = 4$ and $t_1 = t_2 = t_3 = t_4 = [2,2,1]$; or
- $n = 3$, $t_1 = t_2 = [2,2,1]$ and $t_3 = [5]$.

If $G = \mathfrak{A}_5$, Theorem I.11 yields $g_{\hat{X}} = 1$ for signature $(0;2,2,2,2)$ and $g_{\hat{X}} = -7$ for signature $(0;2,2,5)$; so the second case is not possible. Moreover, since the full group of automorphisms of a surface of genus 1 is a solvable group, the non-solvable group $\mathfrak{A}_5$ cannot be a subgroup of it. So $\mathrm{Mon}(f)$ is not $\mathfrak{A}_5$, that proves item (5a).

Theorem IV.8 states that a product $\prod_{i=1}^{n} c_i$ of permutations in D_5 with an odd amount of type $[2,2,1]$ factors is of type $[2,2,1]$, that proves item (5b).

If $\deg(t_1,\dots,t_n) > 8$ and there is an even number of types $[2,2,1]$ in $(t_1,\dots,t_n)$, both $\mathfrak{A}_5$ and D_5 are possible monodromy groups, which proves item (5c). Since now we have two possible monodromy groups, we give an existence proof for both cases separately.

Case $\mathrm{Mon}(f) = \mathfrak{A}_5$. If there is a type $[5]$ in $(t_1,\dots,t_n)$, say t_1, then, modulo re-enumeration, we have $t_2 = t_3 = [2,2,1]$ and $\deg(t_4,\dots,t_n) \geq 4$. According to Theorem IV.5, we can choose a permutation c_i of type t_i for each $i \in \{4,\dots,n\}$ such that $\prod_{i=4}^{n} c_i = (1\,5\,4\,3\,2)$; set $c_1 := (1\,4\,5\,3\,2)$, $c_2 := (1\,2)(3\,5)$ and $c_3 := (1\,5)(2\,3)$, so $\prod_{i=1}^{n} c_i = \mathrm{Id}$. Also, note that $c_1c_2 = (2\,4\,5)$, hence $\langle c_1,\dots,c_n\rangle$ contains only even permutations and of every possible type, so $\langle c_1,\dots,c_n\rangle = \mathfrak{A}_5$. Therefore, $(c_1,\dots,c_n)$ is a generating $(0;5,2,2,m_4,\dots,m_n)$-vector for $\mathfrak{A}_5$.

If there are no types $[5]$ in $(t_1,\dots,t_n)$, then $t_i = [2,2,1]$ for each $i \in \{1,\dots,n\}$, and $n \geq 6$. By Theorem IV.5, we can choose a permutation c_i of type $[2,2,1]$ for each $i \in \{5,\dots,n\}$ such that $\prod_{i=5}^{n} c_i = (1\,5\,4\,3\,2)$; set $c_1 := (1\,5)(2\,3)$, $c_2 := (1\,4)(2\,5)$, $c_3 := (1\,2)(3\,5)$ and $c_4 := (1\,5)(2\,3)$, so $\prod_{i=1}^{n} = \mathrm{Id}$. We also have that $c_1c_2c_3 = (2\,4\,5)$, hence $\langle c_1,\dots,c_n\rangle = \mathfrak{A}_5$ as in the previous case. Therefore, $(c_1,\dots,c_n)$ is a generating $(0;2,\dots,2)$-vector for $\mathfrak{A}_5$.

Case $\mathrm{Mon}(f) \cong D_5$. There are at least two types $[2,2,1]$ in $(t_1,\dots,t_n)$, say t_1 and t_2. By Theorem IV.9 we can choose a permutation c_i of type t_i in D_5 for each $i \in \{3,\dots,n\}$ such that $\prod_{i=3}^{n} c_i = (1\,5\,4\,3\,2)$. Set $c_1 := (1\,3)(4\,5)$ and $c_2 := (1\,2)(3\,5)$ (both permutations belong to D_5), so $\prod_{i=1}^{n} c_i = \mathrm{Id}$ and $\langle c_1,\dots,c_n\rangle = D_5$. Hence, $(c_1,\dots,c_n)$ is a generating $(0;2,2,m_3,\dots,m_n)$-vector of D_5.

ITEM (6). Note that each type [5] permutation in D_5 or $\mathrm{Aff}(\mathbb{F}_5)$ is contained in their subgroup C_5, so neither D_5 nor $\mathrm{Aff}(\mathbb{F}_5)$ can be generated by elements of type [5]; also, each type [5] permutation in $\mathfrak{S}_5$ is contained in $\mathfrak{A}_5$, so $\mathfrak{S}_5$ cannot be generated by elements of type [5]. Therefore, the monodromy group $\mathrm{Mon}(f)$ must be conjugate to C_5 or $\mathfrak{A}_5$. If $n = 2$, then $\mathrm{Mon}(f)$ is generated by a single permutation of type [5], hence $\mathrm{Mon}(f) \cong C_5$. If $n > 2$, then $\mathrm{Mon}(f)$ is conjugate to C_5 or equals $\mathfrak{A}_5$; since now we have two possible monodromy groups, we prove existence for each case separately.

CASE $\mathrm{Mon}(f) \cong C_5$. According to Theorem IV.9, we can choose a permutation c_i of type [5] in C_5 for each $i \in \{2, \dots, n\}$ such that $\prod_{i=1}^{n} c_i = (1\,5\,4\,3\,2)$; set $c_1 := (1\,2\,3\,4\,5)$, so $\prod_{i=1}^{n} c_i = \mathrm{Id}$ and $\langle c_1, \dots, c_n \rangle = C_5$. Therefore, $(c_1, \dots, c_n)$ is a generating $(0; 5, \dots, 5)$-vector of C_5.

CASE $\mathrm{Mon}(f) = \mathfrak{A}_5$. According to Theorem IV.5, we can choose a permutation c_i of type [5] for each $i \in \{3, \dots, n\}$ such that $\prod_{i=3}^{n} c_i = (1\,5\,4\,3\,2)$. Set $c_1 := (1\,3\,2\,5\,4)$ and $c_2 := (1\,3\,5\,4\,2)$, so $\prod_{i=1}^{n} c_i = \mathrm{Id}$. Note that $c_1^2 c_2^2 = (2\,5\,4)$, hence $\langle c_1, \dots, c_n \rangle = \mathfrak{A}_5$. Therefore, $(c_1, \dots, c_n)$ is a generating $(0; 5, \dots, 5)$-vector of $\mathfrak{A}_5$. □

Theorem IV.12 implies directly the following result, which is just a restatement of the converses of items (1) to (6).

Corollary IV.13. *Consider a degree 5 holomorphic map $f\colon X \to \mathbb{P}$. The following statements hold*:

(1) *If* $\mathrm{Mon}(f) \cong C_5$, *then*

$$R_f = \sum_{j=1}^{n_1} 4p_j,$$

where $n_1 \geq 2$ and $p_1, \dots, p_n$ are different points in X.

(2) *If* $\mathrm{Mon}(f) \cong D_5$, *then*

$$R_f = \sum_{j=1}^{n_1} 4p_j + \sum_{j=1}^{n_2} (q_j + r_j),$$

where n_2 is even and positive, and $p_1, \dots, p_{n_1}, q_1, \dots, q_{n_2}, r_1, \dots, r_{n_2}$ are different points in X such that $f(q_j) = f(r_j)$ for each $j \in \{1, \dots, n_2\}$. If $n_1 = 0$, then $n_2 \geq 4$.

(3) *If* $\mathrm{Mon}(f) \cong \mathrm{Aff}(\mathbb{F}_5)$, *then*

$$R_f = \sum_{j=1}^{n_1} 4p_j + \sum_{j=1}^{n_2} (q_j + r_j) + \sum_{j=1}^{n_3} 3s_j,$$

where n_3 is even and positive, and $p_1, \dots, p_{n_1}, q_1, \dots, q_{n_2}, r_1, \dots, r_{n_2}, s_1, \dots, s_{n_3}$ are different points in X such that $f(q_j) = f(r_j)$ for each $j \in \{1, \dots, n_2\}$. If $n_1 = n_2 = 0$, then $n_3 \geq 4$.

(4) *If* $\mathrm{Mon}(f) = \mathfrak{A}_5$, *then*

$$R_f = \sum_{j=1}^{n_1} 4p_j + \sum_{j=1}^{n_2}(q_j + r_j) + \sum_{j=1}^{n_4} 2t_j,$$

where $p_1, \ldots, p_{n_1}, q_1, \ldots, q_{n_2}, r_1, \ldots, r_{n_2}, t_1, \ldots, t_{n_4}$ *are different points in* X *such that* $f(q_j) = f(r_j)$ *for each* $j \in \{1, \ldots, n_2\}$. *Also, we have* $\deg(R_f) \geq 8$, *and if* $n_4 = 0$, *then* $\deg(R_f) > 8$.

(5) *If* $\mathrm{Mon}(f) = \mathfrak{S}_5$, *then*

$$R_f = \sum_{j=1}^{n_1} 4p_j + \sum_{j=1}^{n_2}(q_j + r_j) + \sum_{j=1}^{n_3} 3s_j + \sum_{j=1}^{n_4} 2t_j + \sum_{j=1}^{n_5}(2u_j + v_j) + \sum_{j=1}^{n_6} w_j,$$

where n_3, n_5 *and* n_6 *are even and not all zero, and* $p_1, \ldots, p_{n_1}$, $q_1, \ldots, q_{n_2}$, $r_1, \ldots, r_{n_2}$, $s_1, \ldots, s_{n_3}$, $t_1, \ldots, t_{n_4}$, $u_1, \ldots, u_{n_5}$, $v_1, \ldots, v_{n_5}$, $w_1, \ldots, w_{n_6}$ *are different points in* X *such that* $f(q_j) = f(r_j)$ *for each* $j \in \{1, \ldots, n_2\}$, $f(u_j) = f(v_j)$ *for each* $j \in \{1, \ldots, n_5\}$, $f(t_j) \neq f(w_k)$ *for each pair* (j, k), *and* $f(w_j) \neq f(w_k)$ *for* $j \neq k$. *Also, we have* $\deg(R_f) \geq 8$, *and if* $n_4 = n_5 = n_6 = 0$, *then* $\deg(R_f) > 8$.

CHAPTER V

Decomposition of the Jacobian of a fivefold cover

Following the notation of chapter IV, let $f\colon X \to Y$ be a degree 5 holomorphic map between compact Riemann surfaces with ramification divisor R_f. Theorem IV.6 states that $\mathrm{Mon}(f)$ is conjugate to one of the following groups:

(1) the cyclic group C_5;
(2) the dihedral group D_5;
(3) the affine group $\mathrm{Aff}(\mathbb{F}_5)$;
(4) the alternating group $\mathfrak{A}_5$; or
(5) the symmetric group $\mathfrak{S}_5$.

In this chapter we give, up to isogeny, the group algebra decomposition of $\mathrm{Jac}(\hat{X})$ in terms of the Jacobian and Prym varieties associated to the intermediate coverings of $\hat{f}$ for each possible monodromy group $\mathrm{Mon}(f)$ given by Theorems IV.10 and IV.12. Also, we compute the polarization type of the several abelian varieties involved.

1. Cyclic monodromy group

In this section, we assume that $\mathrm{Mon}(f)$ is cyclic of order 5, that is $\mathrm{Mon}(f) = \mathrm{C}_5$, where $\mathrm{C}_5 = \langle(1\,2\,3\,4\,5)\rangle$, as in chapter IV. Since all the non trivial element of C_5 are in the same rational conjugacy class, there are just two rational irreducible representations of C_5: one of degree 1, the trivial representation, and other of degree 4, the restriction of the standard representation (see [**9**, p. 27]); both representations have Schur index 1. They will be respectively denoted by U and V. Table 9 show the rational character table of C_5.

Moreover, since any complex irreducible representation Galois associated to V is of degree 1 (see [**25**, section 5.1]), Theorem III.6 implies that the group algebra decomposition of $\mathrm{Jac}(X)$ is of the form $\mathrm{Jac}(X) \sim A_1 \times A_2$.

TABLE 9. Rational character table of C_5

	1	4
C_5	Id	$(1\,2\,3\,4\,5)$
U	1	1
V	4	-1

Besides, according to Theorems IV.11 and IV.13, the map f is étale or $R_f = \sum_{j=1}^{n} 4p_j$ with $n \geq 2$; the former case is only possible if $g_Y \geq 1$. Note that in this case f is already Galois because $|\mathrm{Mon}(f)| = \deg f$; since 5 is a prime integer, f has no intermediate coverings.

Theorem V.1. *Let $f\colon X \to Y$ be a degree 5 holomorphic map between compact Riemann surfaces. If $\mathrm{Mon}(f) \cong \mathrm{C}_5$, then*

$$\mathrm{Jac}(\hat{X}) \sim \mathrm{Jac}(Y) \times \mathrm{Prym}(f)$$

is the group algebra decomposition of $\mathrm{Jac}(\hat{X})$. *The dimensions of the involved abelian varieties are*:

- $\dim \mathrm{Jac}(Y) = g_Y$
- $\dim \mathrm{Prym}(f) = 4g_Y - 4 + 2n$

The induced polarization on $\mathrm{Jac}(Y) \times \mathrm{Prym}(f)$ *is of type* $(\underbrace{1,\dots,1}_{3g_Y-3},\underbrace{5,\dots,5}_{2g_Y-1})$ *if f is étale and of type* $(\underbrace{1,\dots,1}_{3g_Y-4+2n},\underbrace{5,\dots,5}_{2g_Y})$ *otherwise.*

Proof. The only proper subgroup of C_5 is the trivial one $\{\mathrm{Id}\}$, but it will be enough: Theorem II.2 implies $\rho_{\{\mathrm{Id}\}} = U \oplus V$ and $\rho_{\mathrm{C}_5} = U$; thereby, Theorem III.9 implies that $\mathrm{Prym}(f) \sim A_2$. Therefore, we have

$$\mathrm{Jac}(\hat{X}) \sim \mathrm{Jac}(Y) \times \mathrm{Prym}(f). \tag{V.1}$$

From Riemann–Hurwitz formula, we get $g_X = 5g_Y - 4 + 2n$. Then

$$\dim \mathrm{Jac}(\hat{X}) = 5g_Y - 4 + 2n$$

and

$$\begin{aligned}\dim \mathrm{Prym}(f) &= \dim \mathrm{Jac}(\hat{X}) - \dim \mathrm{Jac}(Y)\\ &= 4g_Y - 4 + 2n.\end{aligned}$$

Theorem III.1 yields that

$$|\ker f^*| = \begin{cases} 5 & \text{if } n = 0,\\ 1 & \text{if } n \geq 2.\end{cases}$$

Recall that, according to Remark III.1, the isogeny of equation (V.1) is given by inclusions and f^*; moreover, by Theorem III.2, the polarization $(f^*)^*\Theta_{f^*\mathrm{Jac}(Y)}$ is analytically equivalent to $\Theta_Y^{\otimes 5}$, so it is of type $(\underbrace{5,\dots,5}_{g_y})$. Also,

$$\mathrm{K}(\Theta_{\mathrm{Prym}(f)}) = \mathrm{K}(\Theta_{f^*\mathrm{Jac}(Y)}) \cong \frac{(\ker f^*)^{\perp}}{\ker f^*} \cong \begin{cases} \mathrm{C}_5^{2g_Y-2} & \text{if } n = 0,\\ \mathrm{C}_5^{2g_Y} & \text{if } n \geq 2;\end{cases}$$

therefore,

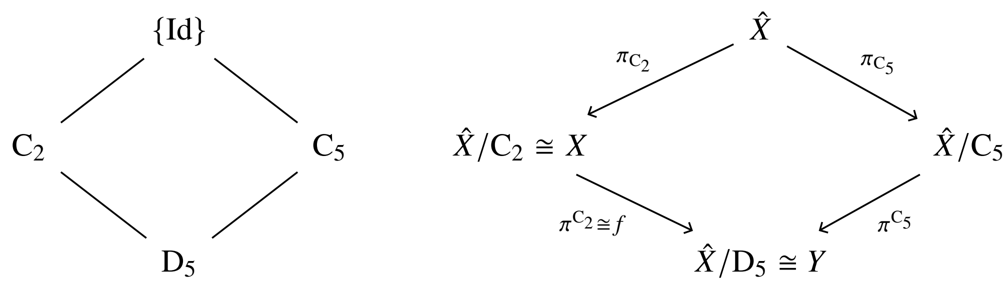
$$\operatorname{type}(\Theta_{\operatorname{Prym}(f)}) = \begin{cases} (\underbrace{1,\dots,1}_{3g_Y-3}\,\underbrace{5,\dots,5}_{g_Y-1}) & \text{if } n = 0, \\ (\underbrace{1,\dots,1}_{3g_Y-4+2n}\,\underbrace{5,\dots,5}_{g_Y}) & \text{if } n \geq 2. \end{cases}$$

Therefore, Theorem III.3 implies the assertion on the type of the polarization induced on $\operatorname{Jac}(Y) \times \operatorname{Prym}(f)$. □

2. Dihedral monodromy group

Now we assume that $\operatorname{Mon}(f)$ is dihedral of order 10, that is $\operatorname{Mon}(f) = \mathrm{D}_5$, where $\mathrm{D}_5 = \langle (1\,2\,3\,4\,5), (2\,5)(3\,4) \rangle$, as in Theorem IV.6. We denote $\langle (1\,2\,3\,4\,5) \rangle$ and $\langle (2\,5)(3\,4) \rangle$ by C_5 and C_2, respectively. The subgroup lattice of D_5 yields intermediate coverings of $\hat{f}$ as in the following commutative diagram:

(V.2)



Since $\operatorname{Stab}_{\mathrm{D}_5}(1) = \mathrm{C}_2$, Theorem I.4 implies that $\hat{X}/\mathrm{C}_2 \cong X$ and $\pi^{\mathrm{C}_2} \cong f$.

According to Theorems IV.11 and IV.13, we have

$$R_f = \sum_{j=1}^{n_1} 4p_j + \sum_{j=1}^{n_2} (q_j + r_j), \tag{V.3}$$

where n_2 is even and $p_1, \dots, p_{n_1}, q_1, \dots, q_{n_2}, r_1, \dots, r_{n_2}$ are different points in X such that $f(q_j) = f(r_j)$; also:

- if $g_Y = 1$, then n_1 and n_2 cannot be both zero; and
- if $g_Y = 0$, then $n_2 \geq 2$ and if $n_1 = 0$, then $n_2 \geq 4$.

Theorem I.14 and equation (V.3) imply that the signature of $\hat{f}$ is

$$(g_Y; \underbrace{2,\dots,2}_{n_2}, \underbrace{5,\dots,5}_{n_1}).$$

TABLE 10. Total ramification of the intermediate coverings of the Galois closure of a convering f with $\mathrm{Mon}(f) \cong \mathrm{D}_5$

H	Genus of $\hat{X}/H$	$\deg(R_{\pi_H})$	$\deg(R_{\pi^H})$
$\{\mathrm{Id}\}$	$10g_Y + 4n_1 + 5n_2/2 - 9$	0	$8n_1 + 5n_2$
C_2	$5g_Y + 2n_1 + n_2 - 4$	n_2	$4n_1 + 2n_2$
C_5	$2g_Y + n_2/2 - 1$	$8n_1$	n_2
D_5	g_Y	$8n_1 + 5n_2$	0

TABLE 11. Complex character table of D_5

	1	5	2	2
D_5	Id	$(2\,5)(3\,4)$	$(1\,2\,3\,4\,5)$	$(1\,3\,5\,2\,4)$
U	1	1	1	1
W	1	-1	1	1
W_2	2	0	$2\cos(2\pi/5)$	$2\cos(4\pi/5)$
W_3	2	0	$2\cos(4\pi/5)$	$2\cos(2\pi/5)$

TABLE 12. Rational character table of D_5

	1	5	4
D_5	Id	$(2\,5)(3\,4)$	$(1\,2\,3\,4\,5)$
U	1	1	1
W	1	-1	1
V	4	0	-1

The genera and total ramification of the several intermediate coverings of $\hat{f}$ are computed through the SageMath implementation of Theorem I.17 and Theorem I.16 (see appendix A) and presented in Table 10.

According to [**25**, section 5.3], there are four complex irreducible representations of D_5: two of degree 1, the trivial denoted U and another one denoted W, and two of degree 2, which we will denote by W_2 and W_3. Table 11 shows the complex character table of D_5.

The representations W_2 and W_3 are clearly not rational, but $W_2 \oplus W_3$ is; furthermore, it is the restriction of the standard representation, so we denote it by V. Moreover, the rational conjugacy classes of D_5 are three: the class of the identity, the class of $(2\,5)(3\,4)$ and the class of $(1\,2\,3\,4\,5)$; so, the three rational irreducible representations of D_5 are U, W and V. Table 12 shows the rational character table of D_5.

Rational irreducible representations of D_5 satisfies the following properties:

- $m_U = m_W = m_V = 1$;

- U and W are complex irreducible representations; and
- V is Galois associated to the complex irreducible representation W_2, which is of degree 2.

Therefore, Theorem III.6 implies that the group algebra decomposition of $\mathrm{Jac}(\hat{X})$ is of the form

$$\mathrm{Jac}(\hat{X}) \sim A_1 \times A_2 \times A_3^2.$$

Theorem V.2. *Let $f\colon X \to Y$ be a degree 5 holomorphic map between compact Riemann surfaces. If $\mathrm{Mon}(f) \cong \mathrm{D}_5$, then*

$$\mathrm{Jac}(\hat{X}) \sim \mathrm{Jac}(Y) \times \mathrm{Prym}(\pi^{\mathrm{C}_5}) \times \mathrm{Prym}(f)^2 \tag{V.4}$$

is the group algebra decomposition of $\mathrm{Jac}(\hat{X})$, where D_5 acts trivially on $\mathrm{Jac}(Y)$, and as multiples of W and V on $\mathrm{Prym}(\pi^{\mathrm{C}_5})$ and $\mathrm{Prym}(f)^2$, respectively. The dimensions of the abelian varieties involved are:

- $\dim \mathrm{Jac}(Y) = g_Y$
- $\dim \mathrm{Prym}(\pi^{\mathrm{C}_5}) = g_Y + n_2/2 - 1$
- $\dim \mathrm{Prym}(f) = 4g_Y + 2n_1 + n_2 - 4$

and the types of the polarizations of the Prym varieties are:

- $\text{type}\,\Theta_{\mathrm{Prym}(\pi^{\mathrm{C}_5})} = \begin{cases} (\underbrace{2,\ldots,2}_{g_Y-1}) & \text{if } n_2 = 0, \\ (\underbrace{1,\ldots,1}_{n_2/2-1},\underbrace{2,\ldots,2}_{g_Y}) & \text{if } n_2 \geq 2; \end{cases}$
- $\text{type}\,\Theta_{\mathrm{Prym}(f)} = (\underbrace{1,\ldots,1}_{3g_Y+2n_1+n_2-4},\underbrace{5,\ldots,5}_{g_Y}).$

Proof. We have that:

- The subgroup C_5 has four elements in the D_5-rational conjugacy class of $(1\,2\,3\,4\,5)$, and the identity.
- The subgroup C_2 has one element in the rational D_5-conjugacy class of $(2\,5)(3\,4)$, and the identity.

Thereby, a direct computation using Theorem II.2 and Table 12 shows that

$$\rho_{\mathrm{C}_5} = U \oplus W,$$
$$\rho_{\mathrm{C}_2} = U \oplus V$$

and

$$\rho_{\mathrm{D}_5} = U.$$

Therefore, following the notation of Theorem III.6, Theorem III.9 implies that

$$A_1 \sim \operatorname{Jac}(Y),$$
$$A_2 \sim \operatorname{Prym}(\pi^{\mathrm{C}_5})$$

and

$$A_3 \sim \operatorname{Prym}(f).$$

The dimension of the several Jacobian and Prym varieties are directly computed from the genera of the corresponding curves in Table 10.

Now we compute the polarization type of the Prym varieties in the decomposition. According to Table 10, we have that π^{C_5} is étale if and only if $n_2 = 0$; besides, the map π^{C_5} is cyclic (has degree 2) whereas f is not even Galois, because C_2 is not normal in D_5. Also, neither π^{C_5} nor f factor non-trivially, because there are no proper subgroups of D_5 that contains C_5 or C_2 but themselves, see Theorem I.15. Hence, by Theorem III.1 we have that

$$|\ker \pi^{\mathrm{C}_5 *}| = \begin{cases} 2 & \text{if } n_2 = 0, \\ 1 & \text{if } n_2 \geq 2; \end{cases}$$
$$|\ker f^*| = 1.$$

Using item (2) of Theorem III.2 and the computations in Table 10, we get the types of $\Theta_{\operatorname{Prym}(\pi^{\mathrm{C}_5})}$ and $\Theta_{\operatorname{Prym}(f)}$. □

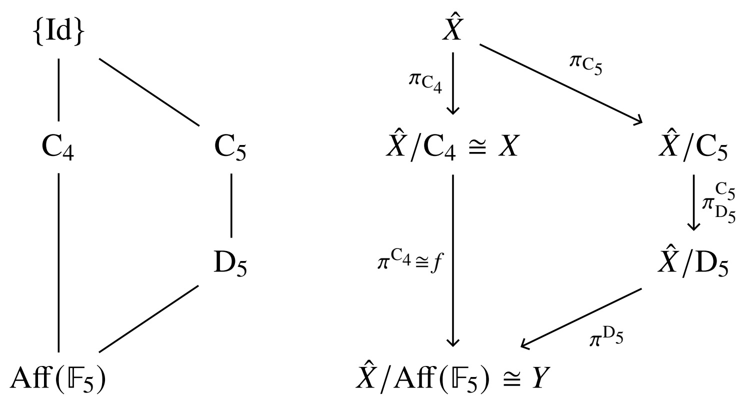
Corollary V.3. *Under the hypotheses of Theorem V.2, if* $Y \cong \mathbb{P}^1$*, then the polarization induced on* $\operatorname{Jac}(Y) \times \operatorname{Prym}(\pi^{\mathrm{C}_5}) \times \operatorname{Prym}(f)^2$ *by isogeny* (V.4) *is of type* $(\underbrace{2,\dots,2}_{4n_1+2n_2-8}, \underbrace{5,\dots,5}_{n_2/2-1})$.

Proof. If $g_Y = 0$, then $\operatorname{Jac}(Y) = \{0\}$, $\operatorname{Jac}(X) = \operatorname{Prym}(f)$ and $\operatorname{Jac}(\hat{X}/\mathrm{C}_5) = \operatorname{Prym}(\pi^{\mathrm{C}_5})$. By item (1) of Theorem III.2, the polarization induced on $\operatorname{Prym}(f)$ by $\Theta_{\hat{X}}$ through $\pi^*_{\mathrm{C}_2}$ is analytically equivalent to $\Theta_X^{\otimes 2}$ and, analogously, the polarization induced on $\operatorname{Prym}(\pi^{\mathrm{C}_5})$ by $\Theta_{\hat{X}}$ through $\pi^*_{\mathrm{C}_5}$ is analytically equivalent to $\Theta^{\otimes 5}_{\hat{X}/\mathrm{C}_5}$. According to Remark III.1, the isogeny (V.4) is given by the natural pullbacks in each component, so the polarization induced on $\operatorname{Jac}(Y) \times \operatorname{Prym}(\pi^{\mathrm{C}_5}) \times \operatorname{Prym}(f)^2$, which with the current restrictions is isogenous to $\operatorname{Prym}(\pi^{\mathrm{C}_5}) \times \operatorname{Prym}(f)^2$, by isogeny (V.4) is of type $(\underbrace{2,\dots,2}_{4n_1+2n_2-8}, \underbrace{5,\dots,5}_{n_2/2-1})$. □

3. Affine monodromy group

Now we assume that $\operatorname{Mon}(f)$ is isomorphic to the group of affine transformations of $\mathbb{F}_5$, which we denote by $\operatorname{Aff}(\mathbb{F}_5)$. As in Theorem IV.6, suppose that $\operatorname{Aff}(\mathbb{F}_5) = \langle (1\,2\,3\,4\,5), (2\,3\,5\,4) \rangle$. Since $\langle (1\,2\,3\,4\,5) \rangle$ and $\langle (2\,3\,5\,4) \rangle$ are cyclic groups of order 5 and 4, respectively, and $\langle (2\,5)(3\,4), (1\,2\,3\,4\,5) \rangle$ is dihedral of order 10; we denote these three groups by C_5, C_4 and D_5, respectively. The subgroup lattice of $\mathrm{Aff}(\mathbb{F}_5)$ yields the coverings described in the following commutative diagram (there are other intermediate coverings, but they will not be used in the Jacobian decomposition):

(V.5)



Since $\mathrm{Stab}_{\mathrm{Aff}(\mathbb{F}_5)}(1) = C_4$, Theorem I.4 implies that $\hat{X}/C_4 \cong X$ and $\pi^{C_4} \cong f$.

According to Theorems IV.11 and IV.13, we have

(V.6) $$R_f = \sum_{j=1}^{n_1} 4p_j + \sum_{j=1}^{n_2} (q_j + r_j) + \sum_{j=1}^{n_3} 3s_j,$$

where n_3 is even and $p_1, \ldots, p_{n_1}, q_1, \ldots, q_{n_2}, r_1, \ldots, r_{n_2}, s_1, \ldots, s_{n_3}$ are different points in X such that $f(q_j) = f(r_j)$; also:

- if $g_Y = 1$, then n_1, n_2 and n_3 cannot be all zero; and
- if $g_Y = 0$, then $n_3 \geq 2$ and if $n_1 = n_2 = 0$, then $n_3 \geq 4$.

Theorem I.14 and equation (V.6) imply that the signature of $\hat{f}$ is

$$(g_Y; \underbrace{2, \ldots, 2}_{n_2}, \underbrace{4, \ldots, 4}_{n_3}, \underbrace{5, \ldots, 5}_{n_1}).$$

The genera and total ramification of the several coverings in diagram V.5 were computed through the SageMath implementation of Theorem I.17 and Theorem I.16 (see appendix A) and are presented in Table 13. The total ramification

(V.7) $$\deg R_{\pi^{C_5}_{D_5}} = 2n_2 + n_3$$

was also computed through that implementation.

There are five complex irreducible representations of $\mathrm{Aff}(\mathbb{F}_5)$:

- Four of degree 1:
  - The trivial, which we will denote by U.
  - The restriction of the alternating representation, which will be denoted by $\tilde{U}$.
  - Two more representations, which are dual to each other and will be denoted by W and W^*, respectively.

TABLE 13. Total ramification of the intermediate coverings of the Galois closure of a convering f with $\mathrm{Mon}(f) \cong \mathrm{Aff}(\mathbb{F}_5)$

H	Genus of $\hat{X}/H$	$\deg(R_{\pi_H})$	$\deg(R_{\pi^H})$
$\{\mathrm{Id}\}$	$20g_Y + 8n_1 + 5n_2 + 15n_3/2 - 19$	0	$16n_1 + 10n_2 + 15n_3$
C_4	$5g_Y + 2n_1 + n_2 + 3n_3/2 - 4$	$2n_2 + 3n_3$	$4n_1 + 2n_2 + 3n_3$
C_5	$4g_Y + n_2 + 3n_3/2 - 3$	$16n_1$	$2n_2 + 3n_3$
D_5	$2g_Y + n_3/2 - 1$	$16n_1 + 10n_2 + 5n_3$	n_3
$\mathrm{Aff}(\mathbb{F}_5)$	g_Y	$16n_1 + 10n_2 + 15n_3$	0

TABLE 14. Complex character table of $\mathrm{Aff}(\mathbb{F}_5)$

	1	4	5	5	5
$\mathrm{Aff}(\mathbb{F}_5)$	Id	$(1\,2\,3\,4\,5)$	$(2\,3\,5\,4)$	$(2\,4\,5\,3)$	$(1\,4)(2\,3)$
U	1	1	1	1	1
$\tilde{U}$	1	1	−1	−1	1
W	1	1	i	−i	−1
W^*	1	1	−i	i	−1
V	4	−1	0	0	0

- One of degree 4, which is the restriction of the standard representation and will be denoted by V.

Table 14 shows the complex character table of $\mathrm{Aff}(\mathbb{F}_5)$.

The representations W and W^* are clearly not rational, but their direct sum $W \oplus W^*$ is. Moreover, the rational conjugacy classes of $\mathrm{Aff}(\mathbb{F}_5)$ are four:

(1) the class of Id;
(2) the class of $(1\,2\,3\,4\,5)$;
(3) the class of $(2\,3\,5\,4)$; and
(4) the class of $(2\,5)(3\,4)$.

Therefore, the four rational irreducible representations of $\mathrm{Aff}(\mathbb{F}_5)$ are U, $\tilde{U}$, $W \oplus W^*$ and V. Table 15 shows the rational character table of $\mathrm{Aff}(\mathbb{F}_5)$. Rational irreducible representations of $\mathrm{Aff}(\mathbb{F}_5)$ satisfies the following properties:

- $m_U = m_{\tilde{U}} = m_{W\oplus W^*} = m_V = 1$;
- U, $\tilde{U}$ and V are complex irreducible representations; and
- $W \oplus W^*$ is Galois associated to the complex irreducible representation W, which is of degree 1.

Therefore, Theorem III.6 implies that the group algebra decomposition of $\mathrm{Jac}(\hat{X})$ is of the form

$$\mathrm{Jac}(\hat{X}) \sim A_1 \times A_2 \times A_3 \times A_4^4.$$

Table 15. Rational character table of $\mathrm{Aff}(\mathbb{F}_5)$

	1	4	10	5
$\mathrm{Aff}(\mathbb{F}_5)$	()	(1 2 3 4 5)	(2 3 5 4)	(1 4)(2 3)
U	1	1	1	1
$\tilde{U}$	1	1	−1	1
$W \oplus W^*$	2	2	0	−2
V	4	−1	0	0

Theorem V.4. *Let $f\colon X \longrightarrow Y$ be a degree 5 holomorphic map between compact Riemann surfaces. If $\mathrm{Mon}(f) \cong \mathrm{Aff}(\mathbb{F}_5)$, then*

$$\mathrm{Jac}(\hat{X}) \sim \mathrm{Jac}(Y) \times \mathrm{Prym}(\pi^{\mathrm{D}_5}) \times \mathrm{Prym}(\pi^{\mathrm{C}_5}_{\mathrm{D}_5}) \times \mathrm{Prym}(f)^4 \tag{V.8}$$

is the group algebra decomposition of $\mathrm{Jac}(\hat{X})$, where $\mathrm{Aff}(\mathbb{F}_5)$ acts trivially on $\mathrm{Jac}(Y)$ and as multiples of $\tilde{U}$, $W \oplus W^*$ and V on $\mathrm{Prym}(\pi^{\mathrm{D}_5})$, $\mathrm{Prym}(\pi^{\mathrm{C}_5}_{\mathrm{D}_5})$ and $\mathrm{Prym}(f)^4$, respectively. The dimensions of the abelian varieties involved are:

- $\dim \mathrm{Jac}(Y) = g_Y$
- $\dim \mathrm{Prym}(\pi^{\mathrm{D}_5}) = g_Y + n_3/2 - 1$
- $\dim \mathrm{Prym}(\pi^{\mathrm{C}_5}_{\mathrm{D}_5}) = 2g_Y + n_2 + n_3 - 2$
- $\dim \mathrm{Prym}(f) = 4g_y + 2n_1 + n_2 + 3n_3/2 - 4$

and the types of the polarizations of the Prym varieties are:

- $\text{type } \Theta_{\mathrm{Prym}(\pi^{\mathrm{D}_5})} = \begin{cases} (\underbrace{2, \dots, 2}_{g_Y - 1}) & \textit{if } n_3 = 0, \\ (\underbrace{1, \dots, 1}_{n_3/2-1}, \underbrace{2, \dots, 2}_{g_Y}) & \textit{if } n_3 \geq 2; \end{cases}$
- $\text{type } \Theta_{\mathrm{Prym}(\pi^{\mathrm{C}_5}_{\mathrm{D}_5})} = \begin{cases} (\underbrace{2, \dots, 2}_{2g_Y - 2}) & \textit{if } n_2 = n_3 = 0, \\ (\underbrace{1, \dots, 1}_{n_2 + n_3/2-1}, \underbrace{2, \dots, 2}_{g_{\hat{X}/\mathrm{D}_5}}) & \textit{if } n_2 + n_3 > 0; \end{cases}$
- $\text{type } \Theta_{\mathrm{Prym}(f)} = (\underbrace{1, \dots, 1}_{u}, \underbrace{5, \dots, 5}_{g_Y})$, *where $u = 3g_Y + 2n_1 + n_2 + 3n_3/2 - 4$.*

Proof. We have that:

- The subgroup D_5 has four elements in the rational $\mathrm{Aff}(\mathbb{F}_5)$-conjugacy class of (1 2 3 4 5), five in the $\mathrm{Aff}(\mathbb{F}_5)$-conjugacy class of (2 3 5 4), and the identity.

- The subgroup C_5 has four elements in the rational $\mathrm{Aff}(\mathbb{F}_5)$-conjugacy class of $(1\,2\,3\,4\,5)$, and the identity.
- The subgroup C_4 has two elements in the rational $\mathrm{Aff}(\mathbb{F}_5)$-conjugacy class of $(2\,3\,5\,4)$, one in the $\mathrm{Aff}(\mathbb{F}_5)$-conjugacy class of $(2\,5)(3\,4)$, and the identity.

Thereby, a direct computation using Theorem II.2 and Table 15 shows that

$$\rho_{D_5} = U \oplus \tilde{U},$$
$$\rho_{C_5} = U \oplus \tilde{U} \oplus (W \oplus W^*)$$

and

$$\rho_{C_4} = U \oplus V.$$

Therefore, following the notation of Theorem III.6, Theorem III.9 implies that,

$$A_1 \sim \mathrm{Jac}(Y),$$
$$A_2 \sim \mathrm{Prym}(\pi^{D_5}),$$
$$A_3 \sim \mathrm{Prym}(\pi^{C_5}_{D_5})$$

and

$$A_4 \sim \mathrm{Prym}(f).$$

The dimension of the several Jacobian and Prym varieties are directly computed from the genera of the corresponding curves in Table 13.

Now we compute the polarization type of the Prym varieties in the decomposition. According to Table 13 and equation (V.7), we have that π^{D_5} and $\pi^{C_5}_{D_5}$ are étale if and only if $n_3 = 0$ and $n_2 = n_3 = 0$, respectively; besides, the maps π^{D_5} and $\pi^{C_5}_{D_5}$ are cyclic (both of them are of degree 2) whereas f is not even Galois (because C_4 is not normal in $\mathrm{Aff}(\mathbb{F}_5)$). Also, neither π^{D_5} nor $\pi^{C_5}_{D_5}$ factor non-trivially, because there are no proper subgroups of $\mathrm{Aff}(\mathbb{F}_5)$ that contains D_5 or C_4 but themselves, see Theorem I.15; analogously, the map f also does not factor non-trivially. Hence, by Theorem III.1 we have that

$$|\ker \pi^{D_5 *}| = \begin{cases} 2 & \text{if } n_3 = 0, \\ 1 & \text{if } n_3 \geq 2; \end{cases}$$
$$|\ker \pi^{C_5 *}_{D_5}| = \begin{cases} 2 & \text{if } n_2 = n_3 = 0, \\ 1 & \text{if } n_2 + n_3 > 0; \end{cases}$$
$$|\ker f^*| = 1.$$

Using item (2) of Theorem III.2 and the computations in Table 10, we get the types of $\Theta_{\mathrm{Prym}(\pi^{D_5})}$, $\Theta_{\mathrm{Prym}(\pi^{C_5}_{D_5})}$ and $\Theta_{\mathrm{Prym}(f)}$. □

Corollary V.5. *Under the hypotheses of Theorem V.4, if* $Y \cong \mathbb{P}^1$ *and* $n_3 = 2$*, then the induced polarization on* $\mathrm{Jac}(Y) \times \mathrm{Prym}(\pi^{\mathrm{D}_5}) \times \mathrm{Prym}(\pi^{\mathrm{C}_5}_{\mathrm{D}_5}) \times \mathrm{Prym}(f)^4$ *by the isogeny* (V.8) *is of type* $(\underbrace{4, \dots, 4}_{8n_1+4n_2-4}, \underbrace{5, \dots, 5}_{n_2})$.

Proof. If $g_Y = 0$, then $\mathrm{Jac}(Y) = \{0\}$, $\mathrm{Jac}(X) = \mathrm{Prym}(f)$ and $\mathrm{Jac}(\hat{X}/\mathrm{D}_5) = \mathrm{Prym}(\pi^{\mathrm{D}_5})$; moreover, if $n_3 = 2$, then $\mathrm{Jac}(\hat{X}/\mathrm{D}_5) = \{0\}$ and $\mathrm{Prym}(\pi^{\mathrm{C}_5}_{\mathrm{D}_5}) = \mathrm{Jac}(\hat{X}/\mathrm{C}_5)$. By Theorem III.2, the polarization induced on $\mathrm{Prym}(f)$ by $\Theta_{\hat{X}}$ through $\pi^*_{\mathrm{C}_4}$ is analytically equivalent to $\Theta_X^{\otimes 4}$ and, analogously, the polarization induced on $\mathrm{Prym}(\pi^{\mathrm{C}_5}_{\mathrm{D}_5})$ by $\Theta_{\hat{X}}$ through $\pi^*_{\mathrm{C}_5}$ is analytically equivalent to $\Theta^{\otimes 5}_{\hat{X}/\mathrm{C}_5}$. According to Remark III.1, the isogeny (V.4) is given by the natural pullbacks in each component, so, by Theorem III.3, the polarization induced on $\mathrm{Jac}(Y) \times \mathrm{Prym}(\pi^{\mathrm{D}_5}) \times \mathrm{Prym}(\pi^{\mathrm{C}_5}_{\mathrm{D}_5}) \times \mathrm{Prym}(f)^4$, which with the given restrictions is isogenous to $\mathrm{Prym}(\pi^{\mathrm{C}_5}_{\mathrm{D}_5}) \times \mathrm{Prym}(f)^4$, by the isogeny (V.4) is of type $(\underbrace{4, \dots, 4}_{8n_1+4n_2-4}, \underbrace{5, \dots, 5}_{n_2})$. □

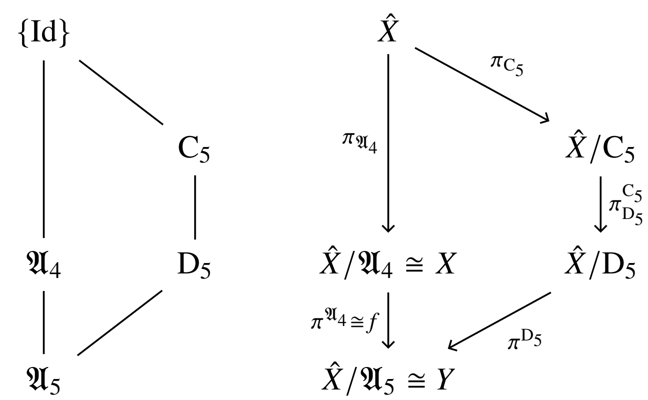
4. Alternating monodromy group

In this section, we assume that $\mathrm{Mon}(f)$ is the alternating group $\mathfrak{A}_5$. In order to archive a cleaner (and more intuitive) notation, we set:

(1) $\mathrm{C}_5 := \langle (1\,2\,3\,4\,5) \rangle$;
(2) $\mathrm{D}_5 := \langle (1\,2\,3\,4\,5), (2\,5)(3\,4) \rangle$; and
(3) $\mathfrak{A}_4 := \langle (2\,3\,4), (3\,4\,5) \rangle$.

The subgroup lattice of $\mathfrak{A}_5$ yields the coverings described in the following commutative diagram (there are other intermediate coverings, but they will not be used in the Jacobian decomposition):

(V.9)



Since $\mathrm{Stab}_{\mathfrak{A}_5}(1) = \mathfrak{A}_4$, Theorem I.4 implies that $\hat{X}/\mathfrak{A}_4 \cong X$ and $\pi^{\mathfrak{A}_4} \cong f$.

TABLE 16. Total ramification of the intermediate coverings of the Galois closure of a covering f with $\mathrm{Mon}(f) \cong \mathfrak{A}_5$

H	Genus of $\hat{X}/H$	$\deg(R_{\pi_H})$	$\deg(R_{\pi^H})$
$\{\mathrm{Id}\}$	$60g_Y + 24n_1 + 15n_2 + 20n_4 - 59$	0	$48n_1 + 30n_2 + 40n_4$
C_5	$12g_Y + 4n_1 + 3n_2 + 4n_4 - 11$	$8n_1$	$8n_1 + 6n_2 + 8n_4$
D_5	$6g_Y + 2n_1 + n_2 + 2n_4 - 5$	$8n_1 + 10n_2$	$4n_1 + 2n_2 + 4n_4$
$\mathfrak{A}_4$	$5g_Y + 2n_1 + n_2 + n_4 - 4$	$6n_2 + 16n_4$	$4n_1 + 2n_2 + 2n_4$
$\mathfrak{A}_5$	g_Y	$48n_1 + 30n_2 + 40n_4$	0

According to Theorems IV.11 and IV.13, we have

$$R_f = \sum_{j=1}^{n_1} 4p_j + \sum_{j=1}^{n_2} (q_j + r_j) + \sum_{j=1}^{n_4} 2t_j, \tag{V.10}$$

where $p_1, \dots, p_{n_1}$, $q_1, \dots, q_{n_2}$, $r_1, \dots, r_{n_2}$, $t_1, \dots, t_{n_3}$ are different points in X such that $f(q_j) = f(r_j)$; also:

- if $g_Y = 1$, then n_1, n_2 and n_4 cannot be all zero, and if $n_1 = n_4 = 0$, then $n_2 \geq 2$; and
- if $g_Y = 0$, then $\deg(R_f) \geq 8$ and if $n_4 = 0$, then $\deg(R_f) > 8$.

Theorem I.14 and equation (V.10) implies that the signature of $\hat{f}$ is

$$(g_Y; \underbrace{2, \dots, 2}_{n_2}, \underbrace{3, \dots, 3}_{n_4}, \underbrace{5, \dots, 5}_{n_1}).$$

The genera and total ramification of the several coverings in diagram V.9 were computed through the SageMath implementation of Theorem I.17 and Theorem I.16 (see appendix A) and are presented in Table 16.

The total ramification

$$\deg R_{\pi_{\mathrm{D}_5}^{\mathrm{C}_5}} = 2n_2 \tag{V.11}$$

was also computed through that implementation; these ramification will be used latter.

According to [9, section 3.1], there are five complex irreducible representations of $\mathfrak{A}_5$:

(1) The trivial, which we will denote by U.
(2) One of degree 4, which is the restriction of the standard representation and which we will denote by V.
(3) One of degree 5, which we will denote by W.
(4) Two of degree 3, which we will denote by W_2 and W_3, that satisfy $W_2 \oplus W_3 = \bigwedge^2 V$.

Table 17 shows the complex character table of $\mathfrak{A}_5$.

The representations W_2 and W_3 are clearly not rational, but their direct sum $\bigwedge^2 V$ is. Moreover, the rational conjugacy classes of $\mathfrak{A}_5$ are four:

TABLE 17. Complex character table of $\mathfrak{A}_5$

	1	20	15	12	12
$\mathfrak{A}_5$	Id	$(1\,2\,3)$	$(1\,2)(3\,4)$	$(1\,2\,3\,4\,5)$	$(2\,1\,3\,4\,5)$
U	1	1	1	1	1
V	4	1	0	-1	-1
W	5	-1	1	0	0
W_2	3	0	-1	$\left(1+\sqrt{5}\right)/2$	$\left(1-\sqrt{5}\right)/2$
W_3	3	0	-1	$\left(1-\sqrt{5}\right)/2$	$\left(1+\sqrt{5}\right)/2$

TABLE 18. Rational character table of $\mathfrak{A}_5$

	1	20	15	24
$\mathfrak{A}_5$	Id	$(1\,2\,3)$	$(1\,2)(3\,4)$	$(1\,2\,3\,4\,5)$
U	1	1	1	1
V	4	1	0	-1
W	5	-1	1	0
$\bigwedge^2 V$	6	0	-2	1

(1) the class of Id;
(2) the class of $(1\,2\,3)$;
(3) the class of $(2\,5)(3\,4)$; and
(4) the class of $(1\,2\,3\,4\,5)$.

Therefore, the four rational irreducible representations of $\mathfrak{A}_5$ are U, V, W and $\bigwedge^2 V$. Table 18 shows the rational character table of $\mathfrak{A}_5$.

Rational irreducible representations of $\mathfrak{A}_5$ satisfies the following properties:

- $m_U = m_V = m_W = m_{\bigwedge^2 V} = 1$;
- U, V and W are complex irreducible representations; and
- $\bigwedge^2 V$ is Galois associated to the complex irreducible representation W_2, which is of degree 3.

Therefore, Theorem III.6 implies that the group algebra decomposition of $\mathrm{Jac}(\hat{X})$ is of the form

$$\mathrm{Jac}(\hat{X}) \sim A_1 \times A_2^4 \times A_3^5 \times A_4^3.$$

Theorem V.6. *Let $f\colon X \to Y$ be a degree 5 holomorphic map between compact Riemann surfaces. If $\mathrm{Mon}(f) \cong \mathfrak{A}_5$, then*

$$\mathrm{Jac}(\hat{X}) \sim \mathrm{Jac}(Y) \times \mathrm{Prym}(f)^4 \times \mathrm{Prym}(\pi^{\mathrm{D}_5})^5 \times \mathrm{Prym}(\pi_{\mathrm{D}_5}^{\mathrm{C}_5})^3$$

is the group algebra decomposition of $\mathrm{Jac}(\hat{X})$*, where* $\mathfrak{A}_5$ *acts trivially on* $\mathrm{Jac}(Y)$ *and as multiples of* V*,* W *and* $\bigwedge^2 V$ *on* $\mathrm{Prym}(f)^4$*,* $\mathrm{Prym}(\pi^{\mathrm{D}_5})^5$ *and* $\mathrm{Prym}(\pi^{\mathrm{C}_5}_{\mathrm{D}_5})^3$*, respectively. The dimensions of the abelian varieties involved are*:

- $\dim \mathrm{Jac}(Y) = g_Y$
- $\dim \mathrm{Prym}(f) = 4g_y + 2n_1 + n_2 + n_4 - 4$
- $\dim \mathrm{Prym}(\pi^{\mathrm{D}_5}) = 5g_Y + 2n_1 + n_2 + 2n_4 - 5$
- $\dim \mathrm{Prym}(\pi^{\mathrm{C}_5}_{\mathrm{D}_5}) = 6g_Y + 2n_1 + 2n_2 + 2n_4 - 6$

and the types of the polarizations of the Prym varieties are:

- type $\Theta_{\mathrm{Prym}(f)} = (\underbrace{1,\dots,1}_{u},\underbrace{5,\dots,5}_{g_Y})$*, where* $u = 3g_Y + 2n_1 + n_2 + n_4 - 4$*;*
- type $\Theta_{\mathrm{Prym}(\pi^{\mathrm{D}_5})} = (\underbrace{1,\dots,1}_{u},\underbrace{6,\dots,6}_{g_Y})$*, where* $u = 4g_Y + 2n_1 + n_2 + 2n_4 - 5$*;*
- type $\Theta_{\mathrm{Prym}(\pi^{\mathrm{C}_5}_{\mathrm{D}_5})} = \begin{cases} (\underbrace{2,\dots,2}_{g_{\hat{X}/\mathrm{D}_5}-1}) & \textit{if } n_2 = n_3 = 0, \\ (\underbrace{1,\dots,1}_{n_2-1},\underbrace{2,\dots,2}_{g_{\hat{X}/\mathrm{D}_5}}) & \textit{if } n_2 + n_3 > 0. \end{cases}$

PROOF. We have that:

(1) The subgroup $\mathfrak{A}_4$ has three elements in the rational $\mathfrak{A}_5$-conjugacy class of $(2\,5)(3\,4)$, eight in the $\mathfrak{A}_5$-conjugacy class of $(2\,3\,4)$, and the identity.
(2) The subgroup D_5 has four elements in the rational $\mathfrak{A}_5$-conjugacy class of $(1\,2\,3\,4\,5)$, five in the $\mathfrak{A}_5$-conjugacy class of $(2\,3\,5\,4)$, and the identity.
(3) The subgroup C_5 has four elements in the rational $\mathfrak{A}_5$-conjugacy class of $(1\,2\,3\,4\,5)$, and the identity.

Thereby, a direct computation using Theorem II.2 and Table 18 shows that

$$\rho_{\mathfrak{A}_4} = U \oplus V,$$
$$\rho_{\mathrm{D}_5} = U \oplus W$$

and

$$\rho_{\mathrm{C}_5} = U \oplus W \oplus \textstyle\bigwedge^2 V.$$

Therefore, following the notation of Theorem III.6, Theorem III.9 implies that,

$$A_1 \sim \mathrm{Jac}(Y),$$
$$A_2 \sim \mathrm{Prym}(f),$$
$$A_3 \sim \mathrm{Prym}(\pi^{\mathrm{D}_5})$$

and

$$A_4 \sim \mathrm{Prym}(\pi^{\mathrm{C}_5}_{\mathrm{D}_5}).$$

The dimension of the several Jacobian and Prym varieties are directly computed from the genera of the corresponding curves in Table 16.

Now we compute the polarization type of the Prym varieties in the decomposition. According to equation (V.11), we have that $\pi^{\mathrm{C}_5}_{\mathrm{D}_5}$ is étale if and only if $n_2 = 0$; besides, the map $\pi^{\mathrm{C}_5}_{\mathrm{D}_5}$ is cyclic because it is of degree 2, whereas neither f nor π^{D_5} are Galois because $\mathfrak{A}_5$ is simple; also, for the same reason, they does not factor by a Galois covering onto Y, see Theorem I.15. Hence, by Theorem III.1 we have that

$$\begin{aligned} |\ker f^*| &= 1,\\ |\ker \pi^{\mathrm{D}_5 *}| &= 1\\ |\ker \pi^{\mathrm{C}_5 *}_{\mathrm{D}_5}| &= \begin{cases} 2 & \text{if } n_2 = 0,\\ 1 & \text{if } n_2 > 0; \end{cases} \end{aligned}$$

Using item (2) of Theorem III.2 and the computations in Table 16, we get the types of $\Theta_{\mathrm{Prym}(f)}$, $\Theta_{\mathrm{Prym}(\pi^{\mathrm{D}_5})}$ and $\Theta_{\mathrm{Prym}(\pi^{\mathrm{C}_5}_{\mathrm{D}_5})}$. □

Corollary V.7. *Under the hypotheses of Theorem V.6, if* $Y \cong \mathbb{P}^1$ *and* (n_1, n_2, n_4) *is* $(2,1,0)$, $(1,1,1)$, $(1,3,0)$ *or* $(0,3,1)$, *then the induced polarization on* $\mathrm{Jac}(Y) \times \mathrm{Prym}(f)^4 \times \mathrm{Prym}(\pi^{\mathrm{D}_5})^5 \times \mathrm{Prym}(\pi^{\mathrm{C}_5}_{\mathrm{D}_5})^3$ *by the isogeny* (V.6) *is of type* $(\underbrace{5,\dots,5}_{3n_2-3}, \underbrace{12,\dots,12}_{4-4n_4})$.

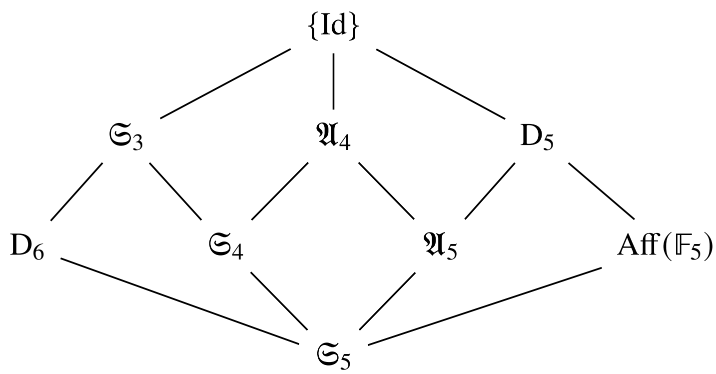
Proof. If $g_Y = 0$, then $\mathrm{Jac}(Y) = \{0\}$, $\mathrm{Jac}(X) = \mathrm{Prym}(f)$ and $\mathrm{Jac}(\hat{X}/\mathrm{D}_5) = \mathrm{Prym}(\pi^{\mathrm{D}_5})$; moreover, if (n_1, n_2, n_4) is any of the listed possibilities, then $\mathrm{Jac}(\hat{X}/\mathrm{D}_5) = \{0\}$ (moreover, those are the only combinations of positive integers that satisfies $\dim \mathrm{Jac}(\hat{X}/\mathrm{D}_5) = 0$ and the conditions imposed on R_f simultaneously) and $\mathrm{Prym}(\pi^{\mathrm{C}_5}_{\mathrm{D}_5}) = \mathrm{Jac}(\hat{X}/\mathrm{C}_5)$. By Theorem III.2, the polarization induced on $\mathrm{Prym}(f)$ by $\Theta_{\hat{X}}$ through $\pi^*_{\mathfrak{A}_4}$ is analytically equivalent to $\Theta_X^{\otimes 12}$ and, analogously, the polarization induced on $\mathrm{Prym}(\pi^{\mathrm{C}_5}_{\mathrm{D}_5})$ by $\Theta_{\hat{X}}$ through $\pi^*_{\mathrm{C}_5}$ is analytically equivalent to $\Theta^{\otimes 5}_{\hat{X}/\mathrm{C}_5}$. According to Remark III.1, the isogeny (V.4) is given by the natural pullbacks in each component, so, by Theorem III.3, the polarization induced on $\mathrm{Jac}(Y) \times \mathrm{Prym}(f)^4 \times \mathrm{Prym}(\pi^{\mathrm{D}_5})^5 \times \mathrm{Prym}(\pi^{\mathrm{C}_5}_{\mathrm{D}_5})^3$, which with the given restrictions is isogenous to $\mathrm{Prym}(f)^4 \times \mathrm{Prym}(\pi^{\mathrm{C}_5}_{\mathrm{D}_5})^3$, by the isogeny (V.4), after reducing the indexes using $2n_1 + n_2 + 2n_4 = 5$, is of type $(\underbrace{5,\dots,5}_{3n_2-3}, \underbrace{5,\dots,5}_{4-4n_4})$. □

5. Symmetric monodromy group

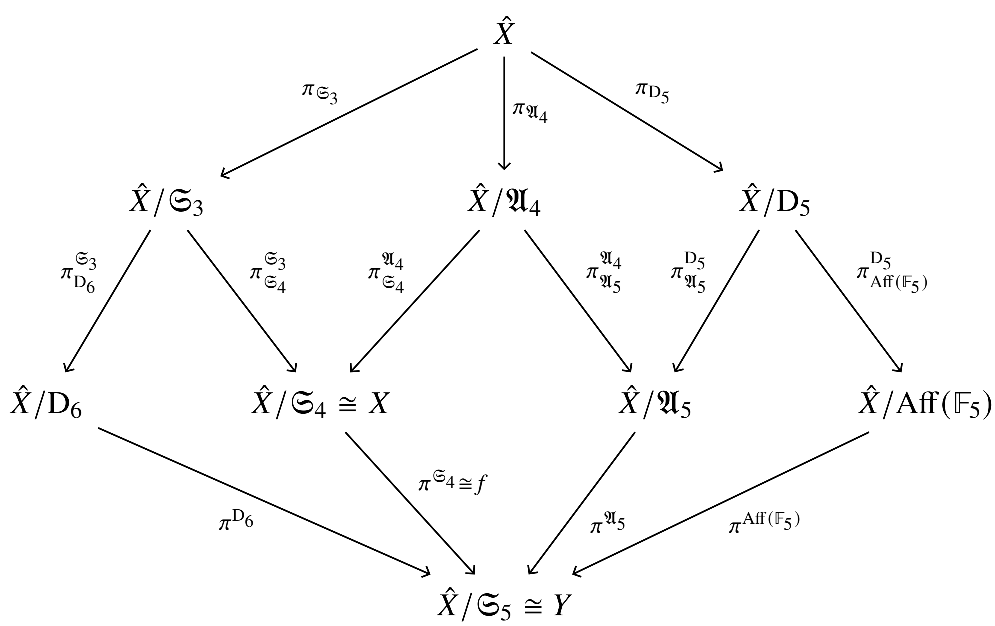
In this section, we assume that $\mathrm{Mon}(f)$ is the whole symmetric group $\mathfrak{S}_5$. In order to archive a cleaner (and more intuitive) notation, we set:

(1) $\mathrm{Aff}(\mathbb{F}_5) \coloneqq \langle (1\,2\,3\,4\,5), (1\,2\,4\,3) \rangle$
(2) $\mathfrak{A}_5 \coloneqq \langle (1\,2\,3\,4\,5), (1\,2\,3) \rangle$
(3) $\mathfrak{S}_4 \coloneqq \langle (2\,3\,4\,5), (2\,3) \rangle$
(4) $\mathrm{D}_6 \coloneqq \langle (1\,2), (3\,4), (3\,4\,5) \rangle$
(5) $\mathrm{D}_5 \coloneqq \langle (1\,2\,3\,4\,5), (2\,5)(3\,4) \rangle$
(6) $\mathfrak{A}_4 \coloneqq \langle (2\,3\,4), (3\,4\,5) \rangle$
(7) $\mathfrak{S}_3 \coloneqq \langle (3\,4\,5), (3\,4) \rangle$

The lattice of subgroups of $\mathfrak{S}_5$ yields the coverings described in the following commutative diagram:



(V.12)



Since $\mathrm{Stab}_{\mathfrak{S}_5}(1) = \mathfrak{S}_4$, Theorem I.4 implies that $\hat{X}/\mathfrak{S}_4 \cong X$ and $\pi^{\mathfrak{S}_4} \cong f$.

According to Theorems IV.11 and IV.13, we have

$$R_f = \sum_{j=1}^{n_1} 4p_j + \sum_{j=1}^{n_2}(q_j + r_j) + \sum_{j=1}^{n_3} 3s_j + \sum_{j=1}^{n_4} 2t_j + \sum_{j=1}^{n_5}(2u_j + v_j) + \sum_{j=1}^{n_6} w_j, \tag{V.13}$$

where n_3, n_5 and n_6 are even and $p_1, \ldots, p_{n_1}, q_1, \ldots, q_{n_2}, r_1, \ldots, r_{n_2}, s_1, \ldots, s_{n_3}, t_1, \ldots, t_{n_4}$, $u_1, \ldots, u_{n_5}$, $v_1, \ldots, v_{n_5}$, $w_1, \ldots, w_{n_6}$ are different points in X such that $f(q_j) = f(r_j)$ for each $j \in \{1, \ldots, n_2\}$, $f(u_j) = f(v_j)$ for each $j \in \{1, \ldots, n_5\}$, $f(t_j) \neq f(w_k)$ for each pair (j, k), and $f(w_j) \neq f(w_k)$ for $j \neq k$; also:

- if $g_Y = 1$, then n_i cannot be zero for all $i \in \{1, \ldots, 6\}$; and
- if $g_Y = 0$, then $\deg(R_f) \geq 8$ and if $n_4 = n_5 = n_6 = 0$, then $\deg(R_f) > 8$.

Theorem I.14 and equation (V.13) implies that the signature of $\hat{f}$ is

$$(g_Y; \underbrace{2, \ldots, 2}_{n_2+n_6}, \underbrace{3, \ldots, 3}_{n_4}, \underbrace{4, \ldots, 4}_{n_3}, \underbrace{5, \ldots, 5}_{n_1}, \underbrace{6, \ldots, 6}_{n_5}).$$

The genera and total ramification of the several coverings in diagram V.12 were computed through the SageMath implementation of Theorem I.17 and Theorem I.16 (see appendix A) and are presented in Table 19.

The following total ramifications where also computed through that implementation:

$$\begin{aligned} \deg R_{\pi_{D_6}^{\mathfrak{S}_3}} &= 2n_2 + n_3 + n_5 + n_6, \\ \deg R_{\pi_{\mathfrak{S}_4}^{\mathfrak{A}_4}} &= n_3 + n_5 + 3n_6 \end{aligned}$$

and

$$\deg R_{\pi_{\mathfrak{A}_5}^{D_5}} = 8n_1 + 4n_2 + 2n_3 + 8n_4 + 4n_5.$$

According to [**9**, section 3.1], there are seven complex irreducible representations of the symmetric group $\mathfrak{S}_5$:

(1) Two of order 1:
  (a) The trivial, which we will denote by U.
  (b) The alternating one, which we will denote by $\tilde{U}$
(2) Two of degree 4:
  (a) The standard representation, which we will denote by V.
  (b) The product $\tilde{U} \otimes V$, which we denote by $\tilde{V}$.
(3) Two of degree 5:
  (a) One that will be denoted by W.
  (b) Another one that corresponds to $\tilde{U} \otimes W$, which we will denote by $\tilde{W}$
(4) One of degree 6, namely $\bigwedge^2 V$.

Table 20 shows the complex character table of $\mathfrak{S}_5$.

The rational conjugacy classes of $\mathfrak{S}_5$ are also seven:

(1) the class of Id;

TABLE 19. Total ramification of the intermediate coverings of the Galois closure of a covering f with $\mathrm{Mon}(f) \cong \mathfrak{S}_5$

H	Genus of $\hat{X}/H$	$\deg(R_{\pi_H})$	$\deg(R_{\pi^H})$
$\{\mathrm{Id}\}$	$120g_Y + 48n_1 + 30n_2 + 45n_3 + 40n_4 + 50n_5 + 30n_6 - 119$	0	$96n_1 + 60n_2 + 90n_3 + 80n_4 + 100n_5 + 60n_6$
$\mathfrak{S}_3$	$20g_Y + 8n_1 + 5n_2 + 15n_3/2 + 6n_4 + 15n_5/2 + 7n_6/2 - 19$	$8n_4 + 10n_5 + 18n_6$	$16n_1 + 10n_2 + 15n_3 + 12n_4 + 15n_5 + 7n_6$
D_5	$12g_Y + 4n_1 + 2n_2 + 4n_3 + 4n_4 + 5n_5 + 3n_6 - 11$	$16n_1 + 20n_2 + 10n_3$	$8n_1 + 4n_2 + 8n_3 + 8n_4 + 10n_5 + 6n_6$
$\mathfrak{A}_4$	$10g_Y + 4n_1 + 2n_2 + 7n_3/2 + 2n_4 + 7n_5/2 + 5n_6/2 - 9$	$12n_2 + 6n_3 + 32n_4 + 16n_5$	$8n_1 + 4n_2 + 7n_3 + 4n_4 + 7n_5 + 5n_6$
D_6	$10g_Y + 4n_1 + 2n_2 + 7n_3/2 + 3n_4 + 7n_5/2 + 3n_6/2 - 9$	$12n_2 + 6n_3 + 8n_4 + 16n_5 + 24n_6$	$8n_1 + 4n_2 + 7n_3 + 6n_4 + 7n_5 + 3n_6$
$\mathrm{Aff}(\mathbb{F}_5)$	$6g_Y + 2n_1 + n_2 + 3n_3/2 + 2n_4 + 5n_5/2 + 3n_6/2 - 5$	$16n_1 + 20n_2 + 30n_3$	$4n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5 + 3n_6$
$\mathfrak{S}_4$	$5g_Y + 2n_1 + n_2 + 3n_3/2 + n_4 + 3n_5/2 + n_6/2 - 4$	$12n_2 + 18n_3 + 32n_4 + 28n_5 + 36n_6$	$4n_1 + 2n_2 + 3n_3 + 2n_4 + 3n_5 + n_6$
$\mathfrak{A}_5$	$2g_Y + n_3/2 + n_5/2 + n_6/2 - 1$	$96n_1 + 60n_2 + 30n_3 + 80n_4 + 40n_5$	$n_3 + n_5 + n_6$

TABLE 20. Complex and rational character table of $\mathfrak{S}_5$

	1	10	20	30	24	15	20
$\mathfrak{A}_5$	Id	(1 2)	(1 2 3)	(1 2 3 4)	(1 2 3 4 5)	(1 2)(3 4)	(1 2)(3 4 5)
U	1	1	1	1	1	1	1
$\tilde{U}$	1	−1	1	−1	1	1	−1
V	4	2	1	0	−1	0	−1
$\tilde{V}$	4	−2	1	0	−1	0	1
$\bigwedge^2 V$	6	0	0	0	1	−2	0
W	5	1	−1	−1	0	1	1
$\tilde{W}$	5	−1	−1	1	0	1	−1

(2) the class of (1 2);
(3) the class of (1 2 3);
(4) the class of (1 2 3 4);
(5) the class of (1 2 3 4 5);
(6) the class of (1 2)(3 4); and
(7) the class of (1 2)(3 4 5).

Therefore, the seven complex irreducible representations of $\mathfrak{S}_5$ are also rational, hence Table 20 is also the rational character table of $\mathfrak{S}_5$. Rational irreducible representations of $\mathfrak{S}_5$ satisfies the following properties:

- all of them have Schur index 1; and
- all of them are complex irreducible representations.

Therefore, Theorem III.6 implies that the group algebra decomposition of $\mathrm{Jac}(\hat{X})$ is of the form

$$\mathrm{Jac}(\hat{X}) \sim A_1 \times A_2 \times A_3^4 \times A_4^4 \times A_5^6 \times A_6^5 \times A_7^5.$$

Theorem V.8. *Let $f\colon X \to Y$ be a degree 5 holomorphic map between compact Riemann surfaces. If $\mathrm{Mon}(f) \cong \mathfrak{S}_5$, then*

$$\mathrm{Jac}(\hat{X}) \sim \mathrm{Jac}(Y) \times \mathrm{Prym}(\pi^{\mathfrak{A}_5}) \times \mathrm{Prym}(f)^4 \times \mathrm{Prym}(\pi^{\mathfrak{A}_4}_{\mathfrak{S}_4}, \pi^{\mathfrak{A}_4}_{\mathfrak{A}_5})^4 \times \mathrm{Prym}(\pi^{\mathfrak{S}_3}_{\mathrm{D}_6}, \pi^{\mathfrak{S}_3}_{\mathfrak{S}_4})^6$$
$$\times \mathrm{Prym}(\pi^{\mathrm{D}_5}_{\mathfrak{A}_5}, \pi^{\mathrm{D}_5}_{\mathrm{Aff}(\mathbb{F}_5)})^5 \times \mathrm{Prym}(\pi^{\mathrm{Aff}(\mathbb{F}_5)})^5$$

is the group algebra decomposition of $\mathrm{Jac}(\hat{X})$*, where $\mathfrak{S}_5$ acts trivially on* $\mathrm{Jac}(Y)$ *and as multiples of $\tilde{U}$, V and $\tilde{V}$, $\bigwedge^2 V$, W and $\tilde{W}$ on* $\mathrm{Prym}(\pi^{\mathfrak{A}_5})$, $\mathrm{Prym}(f)^4$, $\mathrm{Prym}(\pi^{\mathfrak{A}_4}_{\mathfrak{S}_4}, \pi^{\mathfrak{A}_4}_{\mathfrak{A}_5})^4$, $\mathrm{Prym}(\pi^{\mathfrak{S}_3}_{\mathrm{D}_6}, \pi^{\mathfrak{S}_3}_{\mathfrak{S}_4})^6$, $\mathrm{Prym}(\pi^{\mathrm{D}_5}_{\mathfrak{A}_5}, \pi^{\mathrm{D}_5}_{\mathrm{Aff}(\mathbb{F}_5)})^5$ *and* $\mathrm{Prym}(\pi^{\mathrm{Aff}(\mathbb{F}_5)})^5$*, respectively. The dimensions of the abelian varieties involved are*:

- $\dim \mathrm{Jac}(Y) = g_Y$
- $\dim \mathrm{Prym}(\pi^{\mathfrak{A}_5}) = g_Y + n_3/2 + n_5/2 + n_6/2 - 1$
- $\dim \mathrm{Prym}(f) = 4g_Y + 2n_1 + n_2 + 3n_3/2 + n_4 + 3n_5/2 + n_6/2 - 4$

- $\dim \operatorname{Prym}(\pi^{\mathfrak{A}_4}_{\mathfrak{S}_4}, \pi^{\mathfrak{A}_4}_{\mathfrak{A}_5}) = 4g_Y + 2n_1 + n_2 + 3n_3/2 + n_4 + 3n_5/2 + 3n_6/2 - 4$
- $\dim \operatorname{Prym}(\pi^{\mathfrak{S}_3}_{\mathrm{D}_6}, \pi^{\mathfrak{S}_3}_{\mathfrak{S}_4}) = 6g_Y + 2n_1 + 2n_2 + 5n_3/2 + 2n_4/2 + 5n_5/2 + 3n_6/2 - 6$
- $\dim \operatorname{Prym}(\pi^{\mathrm{D}_5}_{\mathfrak{A}_5}, \pi^{\mathrm{D}_5}_{\mathrm{Aff}(\mathbb{F}_5)}) = 5g_Y + 2n_1 + n_2 + 2n_3 + 2n_4 + 2n_5 + n_6 - 5$
- $\dim \operatorname{Prym}(\pi^{\mathrm{Aff}(\mathbb{F}_5)}) = 5g_Y + 2n_1 + n_2 + 3n_3/2 + 2n_4 + 5n_5/2 + 3n_6/2 - 5$

and the types of the polarizations of the Prym varieties are:

- type $\Theta_{\operatorname{Prym}(f)} = (\underbrace{1, \dots, 1}_{u}, \underbrace{5, \dots, 5}_{g_Y})$, *where* $u = 3g_Y + 2n_1 + n_2 + 3n_3/2n_4 + 3n_5/2 + n_6/2 - 4$;
- type $\Theta_{\operatorname{Prym}(\pi^{\mathrm{Aff}(\mathbb{F}_5)})} = (\underbrace{1, \dots, 1}_{v}, \underbrace{6, \dots, 6}_{g_Y})$, *where* $v = 4g_Y + 2n_1 + n_2 + 3n_3/2 + 2n_4 + 5n_2/2 + 3n_6/2 - 5$;
- $$\text{type } \Theta_{\operatorname{Prym}(\pi^{\mathfrak{A}_5})} = \begin{cases} (\underbrace{2, \dots, 2}_{g_Y - 1}) & \textit{if } n_3 = n_5 = n_6 = 0, \\ (\underbrace{1, \dots, 1}_{(n_3+n_5+n_6)/2-1}, \underbrace{2, \dots, 2}_{g_Y}) & \textit{if } n_3 + n_5 + n_6 > 0. \end{cases}$$

Proof. We have that:

(1) The subgroup $\mathrm{Aff}(\mathbb{F}_5)$ has five in the rational $\mathfrak{S}_5$-conjugacy class of $(1\,2)(3\,4)$, ten in the rational $\mathfrak{S}_5$-conjugacy class of $(1\,2\,3\,4)$, four elements in the rational $\mathfrak{S}_5$-conjugacy class of $(1\,2\,3\,4\,5)$, and the identity.
(2) The subgroup $\mathfrak{A}_5$ has twenty elements in the rational $\mathfrak{A}_5$-conjugacy class of $(1\,2\,3)$, fifteen in the rational $\mathfrak{S}_5$-conjugacy class of $(1\,2)(3\,4)$, twenty-four in the rational $\mathfrak{S}_5$-conjugacy class of $(1\,2\,3\,4\,5)$, and the identity.
(3) The subgroup $\mathfrak{S}_4$ has six elements in the rational $\mathfrak{S}_5$-conjugacy class of $(1\,2)$, eight elements in the rational $\mathfrak{S}_5$-conjugacy class of $(1\,2\,3)$, three elements in the rational $\mathfrak{S}_5$-conjugacy class of $(1\,2)(3\,4)$, six elements in the rational $\mathfrak{S}_5$-conjugacy class of $(1\,2\,3\,4)$, and the identity.
(4) The subgroup D_6 has four elements in the rational $\mathfrak{S}_5$-conjugacy class of $(1\,2)$, two elements in the rational $\mathfrak{S}_5$-conjugacy class of $(1\,2\,3)$, three elements in the rational $\mathfrak{S}_5$-conjugacy class of $(1\,2)(3\,4)$, two elements in the rational $\mathfrak{S}_5$-conjugacy class of $(1\,2)(3\,4\,5)$, and the identity.
(5) The subgroup D_5 has five elements in the rational $\mathfrak{S}_5$-conjugacy class of $(1\,2)(3\,4)$, four elements in the rational $\mathfrak{S}_5$-conjugacy class of $(1\,2\,3\,4\,5)$, and the identity.
(6) The subgroup $\mathfrak{A}_4$ has eight elements in the rational $\mathfrak{S}_5$-conjugacy class of $(1\,2\,3)$, three elements in the rational $\mathfrak{S}_5$-conjugacy class of $(1\,2)(3\,4)$, and the identity.
(7) The subgroup $\mathfrak{S}_3$ has three elements in the rational $\mathfrak{S}_5$-conjugacy class of $(1\,2)$, two elements in the rational $\mathfrak{S}_5$-conjugacy class of $(1\,2\,3)$, and the identity.

Thereby, a direct computation using Theorem II.2 and Table 20 shows that

$$\begin{aligned}\rho_{\mathrm{Aff}(\mathbb{F}_5)} &= U \oplus \tilde{W},\\ \rho_{\mathfrak{A}_5} &= U \oplus \tilde{U},\\ \rho_{\mathfrak{S}_4} &= U \oplus V,\\ \rho_{\mathrm{D}_6} &= U \oplus V \oplus W,\\ \rho_{\mathrm{D}_5} &= U \oplus \tilde{U} \oplus W \oplus \tilde{W},\\ \rho_{\mathfrak{A}_4} &= U \oplus \tilde{U} \oplus V \oplus \tilde{V}\end{aligned}$$

and

$$\rho_{\mathfrak{S}_3} = U \oplus 2V \oplus \textstyle\bigwedge^2 V \oplus W.$$

Therefore, following the notation of Theorem III.6, Theorem III.9 implies that

$$\begin{aligned}A_1 &\sim \mathrm{Jac}(Y),\\ A_2 &\sim \mathrm{Prym}(\pi^{\mathfrak{A}_5}),\\ A_3 &\sim \mathrm{Prym}(f)\end{aligned}$$

and

$$A_7 \sim \mathrm{Prym}(\pi^{\mathrm{Aff}(\mathbb{F}_5)}).$$

Also, using Theorems II.3 and III.11, we get

$$\begin{aligned}A_4 &\sim \mathrm{Prym}(\pi^{\mathfrak{A}_4}_{\mathfrak{S}_4}, \pi^{\mathfrak{A}_4}_{\mathfrak{A}_5}),\\ A_5 &\sim \mathrm{Prym}(\pi^{\mathfrak{S}_3}_{\mathrm{D}_6}, \pi^{\mathfrak{S}_3}_{\mathfrak{S}_4})\end{aligned}$$

and

$$A_6 \sim \mathrm{Prym}(\pi^{\mathrm{D}_5}_{\mathfrak{A}_5}, \pi^{\mathrm{D}_5}_{\mathrm{Aff}(\mathbb{F}_5)}).$$

The dimension of the several Jacobian and Prym varieties are directly computed from the genera of the corresponding curves in Table 19 and, for Prym of pairs of coverings, through Theorem III.10.

Now we compute the polarization type of the Prym varieties in the decomposition. According to Table 19, we have that $\pi^{\mathfrak{A}_5}$ is étale if and only if $n_3 = n_5 = n_6 = 0$; besides, the map $\pi^{\mathfrak{A}_5}$ is cyclic (because it is of degree 2) whereas f and $\pi^{\mathrm{Aff}(\mathbb{F}_5)}$ are not (because neither $\mathfrak{S}_4$ nor $\mathrm{Aff}(\mathbb{F}_5)$ are normal in $\mathfrak{S}_5$); also, since neither $\mathfrak{S}_4$ nor $\mathrm{Aff}(\mathbb{F}_5)$ are subgroups of $\mathfrak{A}_5$, which is the only normal subgroup of $\mathfrak{S}_5$, the maps f and $\pi^{\mathrm{Aff}(\mathbb{F}_5)}$ do not factor by a Galois covering onto Y, see Theorem I.15. Hence, by Theorem III.1 we have that

$$\begin{aligned}|\ker f^*| &= 1,\\ |\ker \pi^{\mathrm{Aff}(\mathbb{F}_5)*}| &= 6\end{aligned}$$

and

$$|\ker \pi^{\mathfrak{A}_5*}| = \begin{cases} 2 & \text{if } n_3 = n_5 = n_6 = 0, \\ 1 & \text{if } n_3 + n_5 + n_6 > 0; \end{cases}$$

Using item (2) of Theorem III.2 and the computations in Table 19, we get the types of $\Theta_{\mathrm{Prym}(f)}$, $\Theta_{\mathrm{Prym}(\pi^{\mathrm{Aff}(\mathbb{F}_5)})}$ and $\Theta_{\mathrm{Prym}(\pi^{\mathfrak{A}_5})}$. □

APPENDIX A

SageMath Implementation

```
def RamificationTypes(Group, IncludeTrivial = False):
  if isinstance(Group, sage.libs.gap.element.GapElement):
    G = Group
  else:
    G = Group.gap()
  if IncludeTrivial:
    return [G.ConjugacyClassSubgroups(G.Subgroup(
          [H.Representative()]))
        for H in list(G.RationalClasses())]
  else:
    return [G.ConjugacyClassSubgroups(G.Subgroup(
          [H.Representative()]))
        for H in list(G.RationalClasses())[1:]]

class GaloisCovering:
  def __init__(
      self, Group, QuotientGenus = None,
      GeometricSignature = None):
    if isinstance(Group, sage.libs.gap.element.GapElement):
      self.__Group = Group
    else:
      self.__Group = Group.gap()
    self.__QuotientDegree = self.__Group.Order().sage()
    if self._Group().Order().sage() != 1:
      self._IntermediateCoverings = {K : None for K in list(
        self.__Group.ConjugacyClassesSubgroups())}
    else:
      self._IntermediateCoverings = {self._Group(
        ).ConjugacyClassSubgroups(self._Group()) : self}
    if QuotientGenus is None:
      self.__QuotientGenus = var('g')
    else:
      self.__QuotientGenus = QuotientGenus
    if GeometricSignature is None:
    if GeometricSignature is None:
      if self.__Group.Order().sage() != 1:
        GeometricSignature = list(var(
          ['n' + str(j + 1) for j
```

```
          in range(len(RamificationTypes(self.__Group)))]))
     else:
       GeometricSignature = []
   self.__GeometricSignature = dict(zip(RamificationTypes(
     self.__Group), GeometricSignature))
   self.__Signature = {StabClass.Representative().Order().sage()
           : sum([Num for S, Num
                in self.__GeometricSignature.items()
                if (StabClass
                  .Representative()
                  .Order()
                  .sage()) == (S.Representative()
                       .Order()
                       .sage())])
           for StabClass in self.__GeometricSignature}
   self.__QuotientRamification = {
     Mult : Num * self.__QuotientDegree/Mult
     for Mult, Num in self.__Signature.items()}
   self.__QuotientTotalRamification = sum(
     [(Mult- 1) * Num for Mult, Num
      in self.__QuotientRamification.items()])
   self._InducedDegree = 1
   self._InducedRamification = {}
   self._InducedRamificationData = {}
   self._InducedTotalRamification = 0
   self.__TableOfCoverings = [
     [
       i,
       PermutationGroup(list(
       Class.Representative().GeneratorsOfGroup())),
       PermutationGroup(list(
       Class.Representative().GeneratorsOfGroup()))
       .structure_description(),
       Class.Size(),
       Class.Representative().Order().sage(),
       self._Group().Index(Class.Representative()).sage(),
       '*',
       '*',
       '*']
     for i, Class in enumerate(self._IntermediateCoverings)]

 def IntermediateCovering(self, K = None):
   if K is None:
     K = self._Group()
   Class = self._DetermineClass(K)
   if self._IntermediateCoverings[Class] == None:
     self._IntermediateCoverings[Class] = self if Class == list(
       self._IntermediateCoverings)[-1] else (
```

```
        IntermediateCovering(Class, ParentCovering = self))
      self.__TableOfCoverings[
        list(self._IntermediateCoverings).index(Class)][6:9] = (
          [self.Genus(Class)]
          + list(self.TotalRamifications(Class)))
    return self._IntermediateCoverings[Class]

  def Genus(self, K = None):
    return self.IntermediateCovering(K).__QuotientGenus

  def GeometricSignature(self, K = None):
    return self.IntermediateCovering(K).__GeometricSignature

  def Signature(self, K = None):
    return self.IntermediateCovering(K).__Signature

  def QuotientRamification(self, K = None):
    return self.IntermediateCovering(K).__QuotientRamification

  def InducedRamification(self, K = None, H = None):
    if H is None:
      return self.IntermediateCovering(K)._InducedRamification
    else:
      return self.IntermediateCovering(H).IntermediateCovering(
        self._DetermineClassOfClass(K, H))._InducedRamification

  def InducedRamificationData(self, K = None, H = None):
    if H is None:
      return self.IntermediateCovering(K)._InducedRamificationData
    else:
      return (self.IntermediateCovering(H).IntermediateCovering(
            self._DetermineClassOfClass(K, H))
          ._InducedRamificationData)

  def Ramifications(self, K = None, H = None):
    if H is None:
      return (self.QuotientRamification(K),
          self.InducedRamification(K))
    else:
      return (self.QuotientRamification(K),
          self.InducedRamification(K,H),
          self.InducedRamification(H))

  def QuotientTotalRamification(self, K = None):
    return self.IntermediateCovering(K).__QuotientTotalRamification

  def InducedTotalRamification(self, K = None, H = None):
    if H is None:
```

```
      return (self.IntermediateCovering(K)
        ._InducedTotalRamification)
    else:
      return (self.IntermediateCovering(H)
          .IntermediateCovering(
            self._DetermineClassOfClass(K, H))
          ._InducedTotalRamification)

  def TotalRamifications(self, K = None, H = None):
    if H is None:
      return (self.QuotientTotalRamification(K),
          self.InducedTotalRamification(K))
    else:
      return (self.QuotientTotalRamification(K),
          self.InducedTotalRamification(K, H),
          self.InducedTotalRamification(H))

  def IntermediateCoverings(self, *Show, ComputeAll = False):
    Header = ['#', '$H$', 'Structure',
          '$\\left|\\operatorname{Class}(H)\\right|$',
          '$\\deg \\pi_H $', '$\\deg \\pi^H $', '$g_{X_H}$',
          '$\\left|R_{\\pi_H}\\right|$',
          '$\\left|R_{\\pi^H}\\right|$']
    if Show is not ():
      for Code in Show:
        self.IntermediateCovering(Code)
    return table(
      rows = [
        row for i, row
        in enumerate(self.__TableOfCoverings)
        if i in [list(self._IntermediateCoverings)
             .index(self._DetermineClass(Code))
             for Code in Show]],
      header_row = Header,
      frame = True)
    if ComputeAll is True:
      for Code in range(len(self._IntermediateCoverings)):
        self.IntermediateCovering(Code)
      return table(rows = self.__TableOfCoverings,
             header_row = Header,
             frame = True)

  def InducedIsGalois(self, K, H = None):
    if H is None:
      return self._DetermineClass(K).Size().sage() == 1
    else:
      return self._DetermineClassOfClass(K,H).Size().sage() == 1
```

```
def InducedIsCyclic(self, K, H = None):
  if H is None:
    if self.InducedIsGalois(K):
      return (self._Group().FactorGroup(
            self._DetermineClass(K).Representative())
          .IsCyclic().sage())
    else:
      return False
  else:
    if self.InducedIsGalois(K,H):
      return (self.IntermediateCovering(H)._Group()
          .FactorGroup(self._DetermineClassOfClass(K, H)
                 .Representative())
          .IsCyclic().sage())
  else:
    return False

def InducedAutomorphisms(self, K, H = None):
  if H is None:
    Subgroup = self._DetermineClass(K).Representative()
    return (self._Group().Normalizer(Subgroup)
        .FactorGroup(Subgroup))
  else:
    Subgroup = (self._DetermineClassOfClass(K, H)
          .Representative())
  return (self.IntermediateCovering(K)._Group()
      .Normalizer(Subgroup).FactorGroup(Subgroup))

def _Group(self):
  return self.__Group

def _DetermineClass(self, K):
  if K in self._IntermediateCoverings:
    return K
  elif isinstance(K, sage.rings.integer.Integer):
    return list(self._IntermediateCoverings.keys())[K]
  elif isinstance(K, int):
    return list(self._IntermediateCoverings.keys())[K]
  else:
    try:
      ClassK = (
        self.__Group.ConjugacyClassSubgroups(
          self.__Group.AsSubgroup(K))
        if isinstance(K, sage.libs.gap.element.GapElement)
        else self.__Group.ConjugacyClassSubgroups(
          self.__Group.AsSubgroup(K.gap())))
    except:
      raise Exception(
```

```
          'The argument is not a subgroup of self.__Group')
      for ClassH in self._IntermediateCoverings:
        if ClassK == ClassH:
          return ClassH

  def _DetermineClassOfClass(self, K, H):
    GroupH = self.IntermediateCovering(H)._Group()
    for g in self._DetermineClass(K).AsList():
      if GroupH.IsSubgroup(g):
        return self.IntermediateCovering(H)._DetermineClass(g)
    raise Exception('There are no subgroup in the first class '
            'into a group of the second class')

class IntermediateCovering(GaloisCovering):
  def __init__(self, K, ParentCovering):
    if not isinstance(ParentCovering, GaloisCovering):
      raise Exception(
        'SuperCovering is not a GaloisCovering instance')
    self.__ParentCovering = ParentCovering
    ParentClass = self.__ParentCovering._DetermineClass(K)
    Subgroup = ParentClass.Representative()
    GeometricSignature = dict(zip(
      RamificationTypes(Subgroup),
      [0]*len(RamificationTypes(Subgroup))))
    InducedRamification = {}
    InducedRamificationData = {}
    for StabClass in self.__ParentCovering.GeometricSignature():
      L = [Subgroup.Intersection(Stab)
         for Stab in StabClass.AsList()]
      RT = []
      for StabClassSub in RamificationTypes(
          Subgroup, IncludeTrivial = True):
        Num = L.count(StabClassSub.Representative())
        if Num != 0:
          Images = (
            Num * (self.__ParentCovering
                 .GeometricSignature()[StabClass])
            * (self.__ParentCovering._Group()
               .Normalizer(StabClass.Representative())
               .Index(StabClass.Representative()).sage())
            / (Subgroup.Normalizer(StabClassSub
                         .Representative())
               .Index(StabClassSub.Representative())
               .sage()))
          if StabClassSub in GeometricSignature:
            GeometricSignature[StabClassSub] += Images
          InducedMult = (
            StabClass.Representative()
```

```
          .IndexNC(StabClassSub.Representative())
          .sage())
        RT.append(InducedMult)
        if InducedMult != 1:
          if InducedMult in InducedRamification:
            InducedRamification[InducedMult] += Images
          else:
            InducedRamification[InducedMult] = Images
    if RT and not all(R == 1 for R in RT):
      RT.sort(reverse = True)
      RamType = tuple(RT)
      if RamType in InducedRamificationData:
        InducedRamificationData[RamType] += (
          self.__ParentCovering
          .GeometricSignature()[StabClass])
      else:
        InducedRamificationData[RamType] = (
          self.__ParentCovering
          .GeometricSignature()[StabClass])
  InducedDegree = (self.__ParentCovering._Group()
           .Index(Subgroup).sage())
  InducedTotalRamification = sum(
    [(Mult - 1) * Num for Mult, Num
     in InducedRamification.items()])
  Genus = (
    InducedDegree * (self.__ParentCovering.Genus() - 1) + 1
    + InducedTotalRamification / 2)
  super().__init__(
    Subgroup,
    Genus,
    list(GeometricSignature.values()))
  self._InducedDegree = InducedDegree
  self._InducedRamification = InducedRamification
  self._InducedTotalRamification = InducedTotalRamification
  self._InducedRamificationData = InducedRamificationData
```

Glossary of Symbols

Notation	Description
$\pi_1(X,x)$	Fundamental group of a pointed surface (X,x)
π_G	Quotient map associated to a group G acting on a surface
π^H	Map induced by the subgroup H of a group G acting on a surface
π_N^H	Map induced by a pair of subgroups H and N, with $H \subset N$, of a group G that acts on a surface
f_*	Push-forward induced by a map f
f^*	Pull-back induced by a map f
$\mathrm{N}_G(H)$	Normalizer in a group G of a subgroup H of G
$\mathrm{Aut}(F)$	Group of automorphisms of a covering map F
$\mathfrak{S}_n$	Symmetric group of degree n
$\mathfrak{A}_n$	Alternating group of degree n
C_n	Cyclic group of order n
D_n	Dihedral group of order $2n$
$\mathrm{Aff}(\mathbb{F}_5)$	General affine group of a 1-dimensional affine space over $\mathbb{F}_5$
$\mathrm{Mon}(F)$	Monodromy group of a covering map F
$\mathrm{Stab}_G(1)$	Stabilizer of 1 in a permutation group G
$\hat{F}\colon \hat{X} \to Y$	Galois closure of a covering map $F\colon X \to Y$
$\mathrm{Core}_G(H)$	Core in G of a subgroup H of G
$\lvert G\rvert$	Order of a group G
$[G : H]$	Index of a subgroup H of G
$\deg F$	Degree of a covering map F
$\mathrm{mult}_x(F)$	Multiplicity of a map F at a point x
g_X	Genus of a compact Riemann surface X
R_F	Ramification divisor of a covering F
$\deg(R_F)$	Total ramification of a covering F
$\mathrm{Stab}_G(p)$	Stabilizer of a point $p \in X$ under the action of a group G on X
$G \cdot p$	Orbit of a point $p \in X$ by the action of a group G on X
$\langle g_1, \ldots, g_n \rangle$	Subgroup generated by elements $g_1, \ldots, g_n$ of a group G
g^h	Conjugate ghg^{-1} of g by h
$\mathrm{Class}_G(H)$	Conjugacy class of a subgroup H of a group G

Notation	Description
H^g	Conjugate subgroup gHg^{-1} of a subgroup H by an element g
$\|g\|$	Order of an element g of some group
$[g,h]$	Commutator $g^{-1}h^{-1}gh$ of two elements g and h of some group
$\operatorname{lcm} S$	Least common multiple of finite set of integers S
e	Euler's number
i	Imaginary unit
$\mathrm{Irr}_k(G)$	Irreducible representations of a group G over a field k
$\mathrm{GL}(V)$	General linear group of a vector space V
χ_V	Character of a complex representation $\rho\colon G \to \mathrm{GL}(V)$
m_V	Schur index of a representation $\rho\colon G \to \mathrm{GL}(V)$
$\mathbb{C}_{\mathrm{class}}(G)$	Space of class functions on G
$\langle \chi, \xi \rangle_G$	Inner product between two class functions $\chi, \xi \in \mathbb{C}_{\mathrm{class}}(G)$
$\bar{z}$	Conjugate of a complex number z
$\mathbb{Q}[V]$	Definition field of a representation V
$\mathbb{Q}[\chi_V]$	Field of characters of a representation V
$\mathrm{Gal}(k'/k)$	Galois group of a field extension k'/k
V^σ	Conjugate representation of V by $\sigma \in \mathrm{Gal}(\mathbb{Q}[V]/\mathbb{Q})$
χ_V^σ	Character of th conjugate representation V^σ
Id_G	Identity element of a group G
$\mathbb{Q}[G]$	Rational group algebra of the group G
$\mathrm{tr}(\sigma)$	Trace of σ
$\dim_k(V)$	Dimension of a vector space V over a field k
Ind_H^G	Class function induced on G by a class function of a subgroup H of G
Res_H^G	Restriction to a subgroup H of a G-representation or class function
ρ_H	Representation induced by the trivial H-representation
χ_H	Character of the induced representation ρ_H
$\mathbb{C}_{\mathrm{class}}(G)$	Space of class functions of a group G over $\mathbb{C}$
$\langle \chi, \psi \rangle_G$	Natural inner product between class functions $\chi, \psi \in \mathbb{C}_{\mathrm{class}}(G)$
Fix_H	Set of fixed points by the action of a group H
$(\mathrm{Jac}(Z), \Theta_Z)$	Jacobian variety of a compact Riemann surface Z
ϕ_Θ	Homomorphism associated to the polarization Θ of an abelian variety
$\mathrm{K}(\Theta)$	Kernel of the homomorphism ϕ_Θ
$\mathrm{type}(\Theta)$	Type of the polarization Θ
$\mathrm{End}_{\mathbb{Q}}(J)$	Rational endomorphisms of a Jacobian variety J
$\mathrm{Prym}(f)$	Prym variety of a covering map f
$A[d]$	Set of d-division points of an abelian variety A
Θ_A	Polarization induced on an abelian subvariety A
$\mathrm{Prym}(f_1, f_2)$	Prym variety of a pair of coverings (f_1, f_2)
G'	Derivated subgroup of a group G

Notation	Description
$\mathbb{P}^1$	Riemann sphere

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