FACULTAD DE INGENIERÍA Y CIENCIAS

Departamento de Matemática y Estadística

A REMARK ON Z-ORIENTABILITY OF KLEINIAN GROUPS

Thesis submitted in partial fulfillment of the requirements for the degree of

Doctor en Ciencias mención Matemática

2019

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2010 *Mathematics Subject Classification.* 30F10, 30F20, 30F40

Key words and phrases. Riemann surfaces, algebraic curves, automorphisms, moduli space

This work has been partially supported by
Project FONDECYT 1190001 and the Project Anillo ACT 1415 PIA-CONICYT.

Abstract. The notion of Z-orientability for 2-cell decompositions of a closed Riemann surface was considered by Zapponi to decide if a given Strebel quadratic form has square roots. He also used this notion in the setting of dessins d'enfants to obtain certain unicellular dessins d'enfants in genus zero (a generalization of Leila's flowers) with the property that such a family is Galois-invariant and it contains at least two Galois orbits. Recently, it has been proved that Z-orientability provides a new Galois invariant for dessins d'enfants.

We show how to extend this notion for general Kleinian groups in any dimension. As an application, this notion is used to provide a necessary and sufficient geometrical condition for a non-constant surjective meromorphic map (of finite type) $\varphi : S \to \widehat{\mathbb{C}}$, where S is a connected Riemann surface, to admit an square root, that is, a meromorphic map $\psi : S \to \widehat{\mathbb{C}}$ such that $\varphi = \psi^2$. We also extend this idea to obtain a necessary and sufficient geometrical condition for a non-constant surjective meromorphic map φ (of finite type) to admit an n-root, that is, a meromorphic map $\psi : S \to \widehat{\mathbb{C}}$ such that $\varphi = \psi^n$.

UNIVERSIDAD DE LA FRONTERA
VICERRECTORÍA DE INVESTIGACIÓN Y POSTGRADO

DIRECCIÓN ACADÉMICA DE POSTGRADO

ACTA Nº 4143

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DOCTORADO EN MATEMATICA
DEPTO. MATEMATICA Y ESTADISTICA

Acknowledgement

First of all I want to thank my family Corina, Ian, Sthefany and friends for always supporting me.

Then, in a very special way, I would like to thank Dr. Rubén Hidalgo and Dr. Saúl Quispe for their guidance and friendship.

To the professors Ángel Carocca, Rubí Rodríguez, Elena Olivos, Mauricio Godoy, Carolina Canales, Sebastián Reyes, Matthieu Calvez who shared their knowledge of each class.

To my classmates and friends Adrialy Muci and Eduardo Uribe thank you always.

Finally, I want to thank Rolando Saenz for being the one who managed the agreement between the Central University of Ecuador and the Universidad de la Frontera.

CONTENT

Introduction

The notion of **Z-orientability** was introduced by Zapponi's [**17**] for 2-cell decomposition of an orientable, not necessarily compact, surface. This decomposition is called **Z-orientability** if its faces can be labelled using only two labels such that adjacent faces have different ones. This notion, named by Zapponi as "orientability", was considered to decide if a given Strebel quadratic meromorphic [**16**] on a closed Riemann surface has square roots. In this case, the 2-cell decomposition is the one defined by the graph whose vertices are the zeroes of the form and the edges are non-compact horizontal trajectories.

In [**18**, **19**], Zapponi applied this **Z-orientability** notion in order to produce certain unicellular dessins d'enfants in genus zero (a generalization of Leila's flowers) with the property that such a family is Galois-invariant and it contains at least two Galois orbits.

In [**5**] Girondo, González-Diez, Hidalgo and Jones used *Z-orientability*, to provide a new Galois invariant for dessins d'enfants (that is, it is not decidable from the passport, the mododromy group or the automorphism group of the dessin d'enfant).

In this thesis we show how to extend this notion of **Z-orientability** for general Kleinian groups in any dimension. As an application, first, we see how this notion is used to provide a necessary and sufficient geometrical condition for a non-constant surjective meromorphic map (of finite type) $\varphi : S \to \widehat{\mathbb{C}}$, where S is a connected Riemann surface, to admit an square root, that is, a meromorphic map $\psi : S \to \widehat{\mathbb{C}}$ such that $\varphi = \psi^2$. Then, we extend this idea, to provide a necessary and sufficient geometrical condition for a non-constant surjective meromorphic map (of finite type) φ, with the same conditions, to admit an n-root, that is, a meromorphic map $\psi : S \to \widehat{\mathbb{C}}$ such that $\varphi = \psi^n$.

The thesis is divided into four chapters. Altogether, we review the mathematical concepts with examples that aid the reader in visualizing the results obtained.

In Chapter I, we mention the spaces on which we will work to facilitate the exposition. First, the isometry groups associated with these spaces are mentioned highlighting the properties of isometries; in the case of the hyperbolic plane, the two models of the hyperbolic space and the form of the geodesics for each one are described. Then, the discrete groups of isometries and Kleinian groups definitions and fundamental theorems are indicated. Additionally, the fundamental polyhedron and the Poincaré polyhedron theorem are introduced and examples are provided. Lastly, the notions of real manifold, Riemann surface, function on Riemann surfaces, real orbifold and Riemann orbifold are explained.

In Chapter II, we explain the notion of **Z-orientable** in the quotient geometric orbifold obtained by the action of subgroup of symmetries in terms of a fundamental polyhedron. Then, we prove a theorem that characterizes when an obtained tiling is **Z-orientable**.

In Chapter III, we first consider a meromorphic function between a connected Riemann surface (not necessarily compact) and the Riemann sphere (which must have at least two branch values and to be of finite type) to build a tessellation in S from the inverse image of the arc that crosses all the branch values of the meromorphic function. Finally, we show that the obtained tiling is **Z-orientable** if the meromorphic function between the two surfaces has a square root, and **Z-orientable** does not depend on the chosen arc; thus, it is characteristic of the meromorphic function.

In Chapter IV, we give a generalization of the idea of **Z-orientable** which we call n-**Z-orientable**. By using the results of Chapter 3 we obtain that n-**Z-orientable** is equivalent for a meromorphic function to have an n-root.

CHAPTER I

Preliminaries

I.1. The basic spaces

I.1.1. Euclidian n-space. We use $\mathbb{E}^n$ to denote the **Euclidian n-space**, which corresponds to the space $\mathbb{R}^n$ whose points are denoted by $x = (x_1, \ldots, x_n)$ (or sometimes as (x, t), combining the first $(n-1)$ coordinates as x, and distinguishing the last coordinate as t) together with the local differential metric on $\mathbb{R}^n$

$$ds^2 = dx_1^2 + \ldots + dx_n^2.$$

called **Euclidean metric**.

I.1.2. Hyperbolic n-space. There are two models in the Hyperbolic space, which include:

(1) The upper half space in $\mathbb{R}^n$ denoted by: $\mathbb{H}^n = \{(x,t) \in \mathbb{R}^n : \ t > 0\}$ together with the local differential metric

$$ds^2 = \frac{dx_1^2 + \ldots + dx_n^2}{x_n^2}.$$

called the **Poincaré metric**.

(2) The open unit ball $\mathbb{B}^n = \{x \in \mathbb{R}^n : \ |x| < 1\}$ together with the local differential metric on $\mathbb{B}^n$

$$ds = \frac{2|dx|}{1 - |x|^2}.$$

I.1.3. Spherical n-space. In $\mathbb{R}^n$, the unit sphere is $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n / |x| = 1\}$ which corresponds to the **Spherical space** together with the metric

$$d(P,Q) = \inf\{l(w) : w \text{ is a curve that joins } P \text{ with } Q \text{ in } \mathbb{S}^{n-1}\}$$

where $l(w)$ is the lenght and w is mesured by the Euclidean metric.

I.1.4. Compactification for a point and the stereographic projection. The one point compactification (or compactification of Alexandrow) of $\mathbb{R}^n$, denoted by $\widehat{\mathbb{R}^n}$, is obtained by adding an extra point ∞ called the point at infinity, in other words, $\widehat{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$

We may embed $\widehat{\mathbb{R}^n}$ in $\widehat{\mathbb{R}^{n+1}}$ in the natural way by making the points $x = (x_1, \ldots, x_n)$ and $\widehat{x} = (x_1, \ldots, x_n, 0)$ to correspond (in this case, the point at infinity of the first is sent to the point of infinite of the last). Thus, $\widehat{\mathbb{R}^n}$ is isomorphic to the plane $x_{n+1} = 0$ in $\widehat{\mathbb{R}^{n+1}}$.

The plane $x_{n+1} = 0$ in $\widehat{\mathbb{R}^{n+1}}$ can be mapped in a $1-1$ manner onto the sphere $\mathbb{S}^n$ by the projecting $\widehat{x}$ towards (or away from) e_{n+1} until it meets the sphere $\mathbb{S}^n$ in the unique point $\pi(\widehat{x})$ other than e_{n+1}. This map π is known as the stereographic projection of $\widehat{\mathbb{R}^n}$ onto $\mathbb{S}^n$.

Using the stereographic projection we can transfer the Euclidean metric from $\mathbb{S}^n$ to a metric d on $\widehat{\mathbb{R}^n}$

$$d(x, y) = |\pi(\widehat{x}) - \pi(\widehat{y})| \text{ where } x, y \in \widehat{\mathbb{R}^n}$$

called **Chordal metric**.

Remark 1. A ***k*-sphere** in $\widehat{\mathbb{R}^n}$ is an Euclidean k-dimensional sphere $S(a, r) = \{x \in \mathbb{R}^n : |x - a| = r\}$ in $\mathbb{R}^n$, or the union of point ∞ with a k-dimensional plane $P(a, t) = \{x \in \mathbb{R}^n : \langle x, a\rangle = t\}$ in $\mathbb{R}^n$.

I.2. Groups of isometries

In this section we will describe the isometric groups for Spherical space, Euclidean space and Hyperbolic space considering their two models described in the previous section.

I.2.1. Spherical isometries. The full group of isometries of $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is the orthogonal group $\mathbb{O}^n$.

Example 1. The isometries of $\mathbb{S}^2$ with the spherical metric are the rotations and rotation-reflections.

I.2.2. Euclidean isometries. The full group of isometries of $\mathbb{E}^n$ is generated by orthogonal group $\mathbb{O}^n$ and the Euclidean translations $(x_1, \ldots, x_n) \to (x_1 + a_1, \ldots, x_n + a_n)$.

Example 2. The isometries of $\mathbb{R}^2 = \mathbb{C}$ with the euclidean metric are generated by the rotations, rotation-reflections and translations.

I.2.3. Hyperbolic isometries of the model $\mathbb{H}^n$. The full group of isometries of $\mathbb{H}^n$ is the group $\mathbb{L}^n$ of transformations defined on $\widehat{\mathbb{R}^n}$, generated by restricted to $\mathbb{H}^n$:

i) Translations: $(x_1, \ldots, x_n) \to (x_1 + a_1, \ldots, x_{n-1} + a_{n-1}, x_n)$; $\infty \to \infty$,
ii) Rotations: $(x_1, \ldots, x_n) \to (r(x_1, \ldots x_{n-1}), x_n)$, $r \in \mathbb{O}^{n-1}$; $\infty \to \infty$,
iii) Dilatations: $(x_1, \ldots, x_n) \to (\lambda x_1, \ldots, \lambda x_n)$; $\lambda > 0$, $\infty \to \infty$,
iv) Inversion: $(x_1, \ldots, x_n) \to (x_1, \ldots, x_n)/\|x\|^2$; $0 \to \infty, \infty \to 0$.

Considering the embedding of $\widehat{\mathbb{R}}^{n-1}$ in $\widehat{\mathbb{R}}^{n}$, the group of transformations $\mathbb{L}^n$ is sometimes called the $(n-1)$**-dimensional Möbius group** of transformations in $\widehat{\mathbb{R}}^{n-1}$ and it is also the full group of **conformal maps** (preserving and reverting orientation) of $\widehat{\mathbb{R}}^{n-1}$. We denote by $\mathbb{L}^n_+$ the index two subgroup of orientation preserving elements.

Example 3. We regard $\mathbb{L}^3$ as acting on $\mathbb{E}^2 = \widehat{\mathbb{C}}$, where it acts as a group of all conformal mappings of the Riemann sphere $\widehat{\mathbb{C}}$, including those which reverse orientation. $\mathbb{L}^3_+$ corresponds to the Möbius transformations of $\widehat{\mathbb{C}}$, that is, those of the form:

$$f(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. The ones reversing the orientation are the extended Möbius transformations, that is, the ones of the form:

$$f(z) = \frac{a\bar{z}+b}{c\bar{z}+d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

Every $g \in \mathbb{L}^n_+$ has at least one fixed point in the clausure of $\mathbb{H}^n$. If g has a fixed point in $\mathbb{H}^n$, then it is a elliptic; if g is not elliptic, and has exactly one fixed point on $\partial\mathbb{H}^n$, then it is parabolic; otherwise it is loxodromic.

Remark 2.

(1) The elements of $\mathbb{L}^n$ preserve the family of k-spheres in $\widehat{\mathbb{R}}^{n-1}$, where $k = 0, 1, \ldots, n-2$.
(2) Given two oriented nests of spheres $S_0 \subset S_1 \ldots \subset S_{n-2}$, and $T_0 \subset T_1 \ldots \subset T_{n-2}$ in $\widehat{\mathbb{R}}^{n-1}$, there is an element $g \in \mathbb{L}^n$ mapping one nest onto the other (i.e., $g(S_m) = T_m$, and g maps the positive half of $S_m - S_{m-1}$ onto the positive half plane of $T_m - T_{m-1}$). Further, if x is any point on the positive half of $S_1 - S_0$, then $g(x)$ can be chosen arbitrary on the positive half of $T_1 - T_0$; g is unique once this choice is made.
(3) Let S be a hypersphere in $\widehat{\mathbb{R}}^{n-1}$, then there is a unique reflection $g \in \mathbb{L}^n$, where $g|_S = 1$, and g interchanges the two halves of $\widehat{\mathbb{R}}^{n-1} - S$.
(4) The stabilizer of $\mathbb{H}^n$ in $\mathbb{L}^{n+1}$ is $\mathbb{L}^n$ (Observe that the group $\mathbb{L}^n$ is naturally embedded in $\mathbb{L}^{n+1}$, for each $g \in \mathbb{L}^n$ there is a $g\prime \in \mathbb{L}^{n+1}$ so that $g\prime\widehat{\mathbb{E}}^n = g$ and $g(\mathbb{H}^n) = \mathbb{H}^n$).

I.2.4. Hyperbolic isometries for the ball model $\mathbb{B}^n$. The two standard models for hyperbolic n-space are $\mathbb{H}^n$ and $\mathbb{B}^n$. We know that there is a homeomorphism $q : \mathbb{H}^n \to \mathbb{B}^n$, which happens to the restriction of an element from $\mathbb{L}^{n+1}$.

The group of hyperbolic isometries of $\mathbb{B}^n$ is defined by $\mathbb{P}^n = q\mathbb{L}^n q^{-1}$. We will also sometimes refer to $\mathbb{P}^n$ as $\mathbb{L}^n$ normalized so as to act on $\mathbb{B}^n$. Also we have $\mathbb{P}^n$ as the stability subgroup of $\mathbb{B}^n$ in $\mathbb{L}^{n+1}$ and $\mathbb{P}^n$ acts transitively on $\mathbb{B}^n$.

I.2.5. Geodesics in the hyperbolic space. In $\mathbb{H}^n$, the geodesics lines are the arcs of circles (and Euclidean lines) being orthogonal to the boundary $\mathbb{R}^{n-1}$. Also, the geodesics k-planes are the Euclidean k-spheres (and Euclidean k-planes) being orthogonal to the boundary.

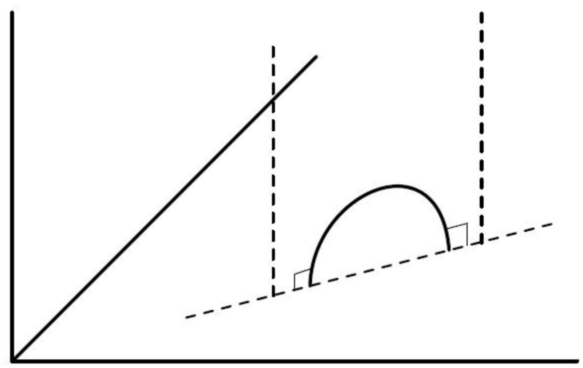

Figure I.1. Geodesic in $\mathbb{H}^3$

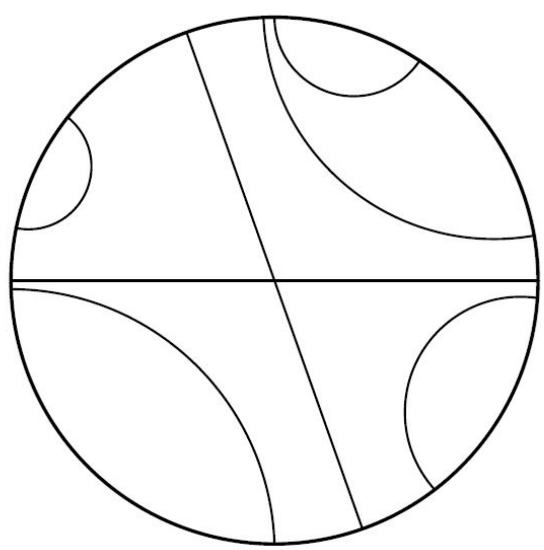

Figure I.2. Geodesic in $\mathbb{B}^2$

At the origin in $\mathbb{B}^n$, the two notions of k-planes and k-spheres coincide. Hence, the planes in hyperbolic geometry have the same incidence relations as those in Euclidean geometry. Also, since the two geometries are conformally the same, any true statement about planes and angles at a point in Euclidean geometry is also true in hyperbolic geometry.

Let $x \neq y$ be points of the hyperbolic space, and let L be the perpendicular bisector of the line segment joining x to y. In this case, L is the geodesic $(n-1)$-plane

$$L = \{z \in \mathbb{H}^n : d(z, x) = d(z, y)\},$$

and it divides the hyperbolic space into two half spaces. If H is the half space containing x, then

$$H = \{z \in \mathbb{H}^n : d(z, x) < d(z, y)\}.$$

In $\mathbb{H}^n$ (or $\mathbb{B}^n$) hyperbolic spheres are Euclidean spheres (in general, with different centers and radius).

I.3. Discrete groups of isometries and Kleinian groups

A natural topology on $\mathbb{L}^n$ is the **compact-open topology**, that is, the topology of uniform convergence on compact subset of $\mathbb{H}^n$, or on $\widehat{\mathbb{R}}^{n-1}$.

Remark 3. Note that $\mathbb{L}^n$ can also be regarded as a matrix group. In fact, for every n, $\mathbb{L}^n_+$ is a canonically isomorphic to a subgroup of index 2 in $\mathrm{SO}(n+1)$ the group of $(n+1)\times(n+1)$ matrices, with real entries and determinant 1, which keep invariant the form $x_1^2 + x_2^2 + \ldots + x_n^2 - x_{n+1}^2$.

The different views of $\mathbb{L}^n$, as acting on $\mathbb{H}^n$, or $\widehat{\mathbb{R}}^{n-1}$, or as matrix group, yield equivalent topologies on $\mathbb{L}^n$.

Let (g_m) be a sequence of elements of $\mathbb{L}^n$. Then, $g_m \to g$ uniformly on compact subsets of $\mathbb{H}^n$ if and only if $g_m \to g$ uniformly on compact subsets of $\widehat{\mathbb{R}}^{n-1}$ if and only if $g_m \to g$ uniformly on $\widehat{\mathbb{E}}^{n-1}$.

Let $\mathbb{X}$ be one of the spaces $\mathbb{H}^n$ (or, equivalently $\mathbb{B}^n$), $\mathbb{S}^n$ (or, equivalently $\widehat{\mathbb{E}}^{n-1}$), or $\widehat{\mathbb{E}}^n$, and $\mathbb{G}$ be the group of isometries of $\mathbb{X}$.

Theorem 1. [**11**] *Let x be a point of $\mathbb{X}$, and let G be a subgroup of $\mathbb{G}$. Then G acts discontinuously at x if and only if G is a discrete subgroup of $\mathbb{G}$.*

Corollary 1. [**11**] *G acts discontinuously at some point of $\mathbb{X}$ if and only if G acts discontinuously at every point of $\mathbb{X}$.*

Remark 4.

(1) If G is a discrete group of $\mathbb{G}$, then for every $x \in \mathbb{X}$, and for every $\rho > 0$, the ball of radius ρ about x contains only finitely many translates of x.
(2) If G is a discrete group of $\mathbb{G}$, and $x \in \mathbb{X}$. The $Stab_G(x)$ is finite, and there is a number $\rho > 0$ so that the ball of radius ρ about x is precisely invariant under $Stab_G(x)$.
(3) Note that every subgroup of $\mathbb{L}^n$ acting discontinuously on $\widehat{\mathbb{R}}^{n-1}$ is discrete. But the converse may fail. For instance, $\mathrm{PSL}_2(\mathbb{Z}[i])$ is a subgroup of $\mathbb{L}^3$ is discrete but its limit set is all of the $\widehat{\mathbb{R}}^2$.

Definition 1 (Kleinian groups)**.** *Let G be a subgroup of $\mathbb{L}^{n+1}$ and let $G^+ = G \cap \mathbb{L}_+^{n+1}$.*

(1) We will denote by $\Omega(G) \subset \widehat{\mathbb{R}}^n$ the set of points where G acts discontinuously, called the **region of discontinuity** *of G. Its complement in $\widehat{\mathbb{R}}^n$ is called its* **limit set** *and denoted by $\Lambda(G)$.*
(2) If $G = G^+$ acts discontinuously at some point of $\widehat{\mathbb{R}}^n$, that is, $\Omega(G) \neq \emptyset$, then we will say that G is a n**-dimensional Kleinian group**.
(3) If $G \neq G^+$ acts discontinuously at some point of $\widehat{\mathbb{R}}^n$, then we will say that G is a n**-dimensional extend Kleinian group**.
(4) If $G < \mathbb{L}^3$ is a 2-dimensional (extended) Kleinian group, then we will say that G is a planar (extended) Kleinian group. In case of larger dimensions, we will speak of (extended) **Kleinian space groups**.

Remark 5. The n-dimensional (extended) Kleinian group is necessarily discrete and every discrete subgroup of $\mathbb{L}^{n+1}$ is necessarily a (extended) Kleinian $(n+1)$-dimensional group. Moreover, if $G < \mathbb{L}^{n+1}$ is such that $G \neq G^+$, then G is an extended Kleinian group if and only if G^+ is a Kleinian group.

Example 4. A (extended) Kleinian group $G < \mathbb{L}^{n+1}$ that leaves invariant a generalized ball $B \subset \widehat{\mathbb{R}}^n$ is called a (extended) **Fuchsian group**.

I.4. Fundamental polyhedrons

We continue the notation of the preceding section. Let $\mathbb{X}$ be one of the spaces $\mathbb{H}^n$ (or $\mathbb{B}^n$), $\mathbb{S}^n$ (or $\widehat{\mathbb{E}}^{n-1}$), or $\widehat{\mathbb{E}}^n$, and let $\mathbb{G}$ be its group of isometries.

A hyperplane in $\mathbb{X}$ divides it into two half spaces.

A (convex) polyhedron D in $\mathbb{X}$ is the intersection of countably many open half-spaces, where only finitely many of the hyperplanes defining these half-spaces meet any compact subset of $\mathbb{X}$. The closure $\overline{D}$ of D has a natural cell decomposition given by the intersections of the defining hyperplanes. The k-cells in this decomposition are called the k-faces of D, or of $\overline{D}$. Also, the codimension one faces are called sides, and the codimension two faces are called edges. Each edge lies in the intersection of exactly two sides.

A polyhedron in dimension two is called a polygon, in this case, the codimension two faces are usually called vertices.

Definition 2. *Let G be a discrete subgroup of $\mathbb{G}$. A polyhedron D is a* **fundamental polyhedron** *for G if the following hold.*

1. *For every non-trivial $g \in G$, $g(D) \cap D = \emptyset$.*
2. *For every $x \in \mathbb{X}$, there is a $g \in G$, with $g(x) \in \overline{D}$.*
3. *The sides of D are paired by elements of G, that is, for every side s there is a side s' (it might not be different), and there is an element $g_s \in G$, with $g_s(s) = s'$. These satisfy the conditions: $g_{s'} = g_s^{-1}$, and $(s')' = s$. The element g_s is called a side pairing transformation.*
4. *Any compact set meets only finitely many G-translates of D.*

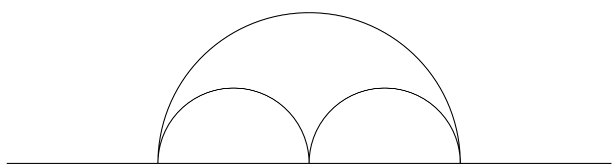

FIGURE I.3. Fundamental polygon

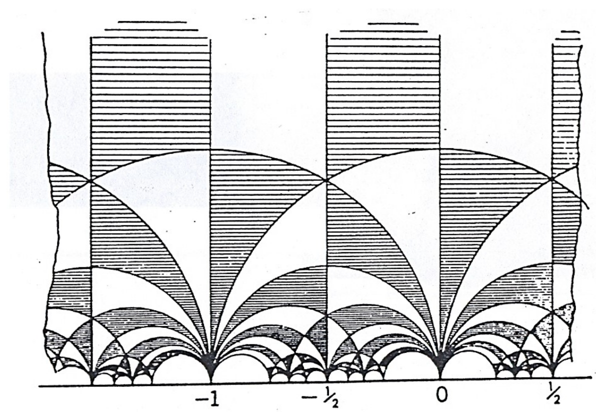

FIGURE I.4. Tesselation of the hyperbolic plane by the modular group

If D is a fundamental polyhedron, then the identification of the sides induce an equivalence relation on $\overline{D}$, that is: $x \sim y$ if there is a finite collection of side pairing transformations $g_1, \ldots, g_r$ with $g_r \circ \cdots g_2 \circ g_1(x) = y$. Condition (1) then says that no two points in

D are equivalent. A consequence of condition (4) is that each point of $\overline{D}$ is equivalent to at most finitely many other points of $\overline{D}$. This equivalence relation also defines an equivalence relation on the edges of D.

I.5. Poincarés polyhedrom theorem

We continue the notation of the preceding section: Let $\mathbb{X}$ be one of the spaces $\mathbb{H}^n$ (or $\mathbb{B}^n$), $\mathbb{S}^n$ (or $\widehat{\mathbb{R}}^{n-1}$), or $\widehat{\mathbb{R}}^n$, and $\mathbb{G}$ is its group of isometries of $\mathbb{X}$. We also assume that $n \geq 2$.

The first condition is that the sides of D are paired by elements of G, that is, we assume that for each side s of D, there is a side s', not necessarily distinct from s, and there is an element $g_s \in G$, called a side pairing transformation, satisfying the following conditions.

(1) $g_s(s) = s'$.

(2) $g_{s'} = g_s^{-1}$.

(3) $g_s(D) \cap D = \emptyset$.

The side pairing transformations induce an equivalence relation on $\overline{D}$. Let D^* be the space of equivalence classes. We should require the following finiteness property on the equivalence classes. s

(4) For every point $z \in D^*$, $p^{-1}(z)$ is a finite set.

For each edge $e = e_1$, let $e_1, \ldots, e_k$ be the ordered set of edges in the cycle containing e, and let $g_1, \ldots, g_k$ be corresponding side pairing transformations. Then the cycle transformation $h = h(e) = g_k \circ \ldots \circ g_1$ keeps e invariant. We require the following finite order condition on these cycles.

(5) For each edge e, there is a positive integer t so that $h^t = 1$.

We let $\alpha(e)$ be the angle, measured from inside D, at the edge e. We also require

(6)
$$\sum_{m=1}^{k} \alpha(e_m) = \frac{2\pi}{t}$$

The space $\mathbb{X}$ has a Riemannian metric on it in which G acts as a group of isometries; we can project this infinitesimal metric to $Z = \mathbb{X}/G$. This distance is defined to be infimum of the lengths of paths connecting z to z'. Equivalently, we can use the natural projection $p : \mathbb{X} \longrightarrow Z$, and define $d(z, z') = \inf d(x, x')$, where $p(x) = z$, and $p(x') = z'$.

We can reconstruct this distance function in $\overline{D}$ as follows. Let x and x' be points of $\overline{D}$, where $p(x) = z$, and $p(x') = z'$. Then $d(z, z') = \inf \sum d(x, x')$, where the infimum is taken over all finite sets of points $\{x_1, x_1', \ldots, x_j, x_j'\}$ in $\overline{D}$, with $p(x_1) = z, p(x_m') = p(x_{m+1}')$, and $p(x_j') = z$. We should require the following completeness property.

(7) D^* , with the above metric, is complete.

Theorem 2 (Poicaré polyhedron theorem [11]). *Let D be a polyhedron with side pairing transformations satisfying conditions (1) through (7). Then G, the group generated by the side pairing transformations is discrete, D is a fundamental polyhedron for G, and the reflection relations and cycle relations form a complete set of relations for G.*

Remark 6.

(1) In dimension 2, each edge is just a point, so condition (4) is automatically satisfied if each cycle of edges is finite. Observe that condition (5) is a consequence of (6).
(2) If D is relatively compact in $\mathbb{X}$, and D, with side pairing transformations, satisfies conditions (1) through (4), then (7) is also satisfied.
(3) Assume that the finite sided polyhedrom $D \in \mathbb{H}^n$ satisfies (1) − (4). Condition (7) is also satisfied if and only if every infinite cycle transformation at every infinite edge is parabolic.

Example 5. Let $C_1, C_1', \ldots, C_k, C_k'$ be $2k$ disjoint $(n-2)$-spheres in $\mathbb{R}^{n-1}$ with a common exterior. For each $m \in \{1, \ldots, k\}$, let g_m be an element of $\mathbb{L}^n$ mapping C_m onto C_m', where g_m maps the inside of C_m onto the out side of C_m'. From the Poincaré theorem in $\mathbb{H}^n$, $G = \langle g_1, \ldots, g_k \rangle$ in discrete and free on these k generators. If all the elements $g_m \in \mathbb{L}_+^n$, then $G = G^+$ and it is called a $(n-1)$-dimensional classical Schottky group of rank k.

If for certain m we permit C_m to be tangent to C_m', and we require the corresponding g_m to be parabolic (necessarily with fixed point at the point of tangency), then we say that G is a noded Schottky group.

Example 6. The Picard group $\mathrm{PSL}(2, \mathbb{Z}[i])$ is the subgroup of $\mathbb{M}$ consisting of all unimodular matrices whose entries are Gaussian integers (i.e. complex numbers of the form $a + ib, a, b \in \mathbb{Z}$). The polyhedron P, formed by the (hyperbolic) lines whose boundaries on the sphere at infinity are the following: $\{z : |z| = 1\}$, $\{z : Re(z) = -\frac{1}{2}\}$, $\{z : Re(z) = \frac{1}{2}\}$, $\{z : \mathrm{Im}(z) = \frac{1}{2}\}$, and $\{z : \mathrm{Im}(z) = 0\}$, is a fundamental polyhedron for $\mathrm{PSL}(2, \mathbb{Z}[i])$, whose side pairing are $z \mapsto -z$, $z \mapsto z + 1$, $z \mapsto -1/z$ and $z \mapsto -z + i$. In particular, this group is generated by them.

I.6. Riemann surfaces

I.6.1. Definition of topological real manifolds.

Definition 3 (Topological real manifold).

(1) *An* **n-dimensional topological real structure** *on a connected, Hausdorff and second countable topological space M is a maximal collection*

$$\mathcal{A} = \{(U_\alpha, \phi_\alpha) : \alpha \in I\},$$

verifying the following properties:

(a) *Each U_α is an open set of M.*

(b) *$M = \bigcup_{\alpha \in I} U_\alpha$.*

(c) *Each ϕ_α is a homeomorphism between U_α and an open set in $\mathbb{R}^n$. We say that ϕ_α is a* **chart local** *of M.*

(2) *An* **n-dimensional topological real manifold** *is a second countable connected Hausdorff topological space together with an n-dimensional topological real structure.*

Remark 7. Note, in the above definition, that if $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a homeomorphism, are the transition maps or coordinate transformations.

I.6.2. Definition of Riemann surfaces.

Definition 4 (Riemann surface)**.** *A* ***Riemann surface*** *is a second countable, connected and Hausdorff topological space S together with a* ***complex structure****, that is, a maximal collection*

$$\mathcal{A} = \{(U_\alpha, \phi_\alpha) : \alpha \in I\},$$

verifying the following properties:

(1) *Each U_α is an open set of S;*

(2) *$S = \bigcup_{\alpha \in I} U_\alpha$;*

(3) *Each ϕ_α is a homeomorphism between U_α and a open set of complex plane $\mathbb{C}$.*

(4) *If $U_\alpha \cap U_\beta \neq \emptyset$, then*

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a biholomorphism.

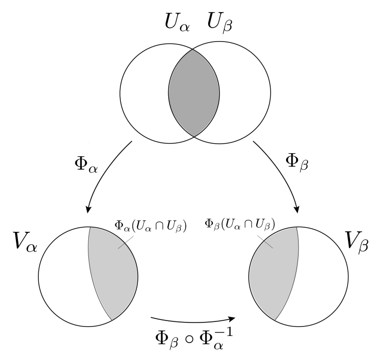

FIGURE I.5. Compatibility

Remark 8.

(1) The collection $\mathcal{A}$ is also called the **Riemann surface structure** and each ϕ_α is called a **local chart** of this structure.
(2) A Riemann surface is a orientable surface, so if we have a compact Riemann surface, according to the classification of compact orientable surfaces, each of these is a g-holed torus for some unique integer $g \geq 0$. This integer g is called the **genus** of Riemann surface.

Below we present some examples of Riemann surfaces.

Example 7. The connected open sets of the complex plane are Riemann surfaces, provided with the identity as the only local chart, in particular the complex plane $\mathbb{C}$, the unitary disc $\mathbb{D} = \{z \in \mathbb{C} : | z |< 1\}$, and the upper half-plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$ are Riemann surfaces. Moreover, each connected open subset of a Riemann surface is a Riemann surface.

Example 8. Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the one-point compactification of the complex plane. The local charts

$$\left\{(\phi_1 : \mathbb{C} \to \mathbb{C} : z \mapsto z)\,,\ \left(\phi_2 : \widehat{\mathbb{C}} - \{0\} \to \mathbb{C} : z \mapsto \frac{1}{z}\right)\right\}$$

makes it a Riemann surface, called the Riemann sphere.

Example 9. Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the one-point compactification of the complex plane. The local charts

$$\left\{(\phi_1 : \mathbb{C} \to \mathbb{C} : z \mapsto \overline{z}),\ \left(\phi_2 : \widehat{\mathbb{C}} - \{0\} \to \mathbb{C} : z \mapsto \frac{1}{\overline{z}}\right)\right\}$$

makes it also a Riemann surface. This is called the conjugated of the Riemann sphere.

Remark 9. Regarding the last two examples presented above, if we are given a Riemann surface S with local charts $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$, then the new local charts $\overline{\mathcal{A}} = \{(U_\alpha, J \circ \phi_\alpha)\}$, where $J(z) = \overline{z}$, produces also a Riemann surface structure, denoted by $\overline{S}$ and called the conjugated of S.

I.6.3. Function on Riemann surfaces.

Definition 5 (Holomorphic and anti-holomorphic functions)**.** *Let $f : S_1 \to S_2$ a function between two Riemann surfaces. We say that the function f is **holomorphic** (respectively **anti-holomorphic**) if for each $p \in S_1$, there are local charts $\phi_1 : U_1 \to V_1$ for S_1 and $\phi_2 : U_2 \to V_2$ for S_2, such that $p \in U_1$, $f(U_1) \subset U_2$ and $\phi_2 \circ f \circ \phi_1^{-1} : V_1 \subset \mathbb{C} \to \mathbb{C}$ is holomorphic (respectively anti-holomorphic) in the usual sense.*

Definition 6 (Biholomorphism and anti-biholomorphism)**.** *Let $f : S_1 \to S_2$ a holomorphic function (respectively anti-holomorphic) and f is a bijection, we call the function f a **biholomorphism** (respectively **anti-biholomorphism**).*

Definition 7 (Holomorphic automorphism and anti-holormorphic automorphism)**.** *Let $f : S \to S$ a biholomorphism (respectively anti-biholomorphism), we call to the function f a **holomorphic automorphism** (respectively, **anti-holomorphic automorphism**). We denote with $Aut^+(S)$ (respectively $Aut(S)$) to the group (with the composition operation) of holomorphic automorphism of S (respectively to the group of the holomorphic automorphism and anti-holomorphic automorphism).*

Example 10. The group of holomorphic automorphisms of the upper half space $\mathbb{H}^2$ is the group $Aut^+(\mathbb{H}^2) = PSL(2, \mathbb{R})$. The anti-holomorphic automorphisms are obtained as composition of a element of $Aut^+(\mathbb{H}^2)$ with $z \mapsto -\overline{z}$.

The Möbius transformation

$$T(z) = \frac{-z+i}{z+i}$$

induces an biholomorphism from $\mathbb{H}^2$ to $\mathbb{B}^2$. Thus, the group of holomorphic automorphisms of the unitary disk $\mathbb{B}^2$ is the group:

$$Aut^+(\mathbb{B}^2) = \left\{ z \mapsto \frac{az+b}{\overline{b}z+\overline{a}} : \ a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1 \right\}.$$

Example 11. The group of holomorphic automorphisms of the complex plane $\mathbb{C}$ is the group:

$$Aut^+(\mathbb{C}) = \{z \mapsto az + b : \ a, b \in \mathbb{C}, a \neq 0\}.$$

The anti-holomorphic automorphisms are obtained as composition of a element of $Aut^+(\mathbb{C})$ with the complex conjugation.

Example 12. The holomorphic automorphisms of the Riemann sphere $\widehat{\mathbb{C}}$ are the Möbius transformations. The anti-holomorphic automorphisms of the Riemann sphere $\widehat{\mathbb{C}}$ are obtained as composition of a Möbius transformation with the complex conjugation (these are called extended Möbius transformations).

I.7. Real orbifold

Definition 8 (Real orbifold)**.** *An n-**dimensional real orbifold** consists of a second countable connected Hausdorff topological space X (called the **underlying topological space of the orbifold**) and of a collection*

$$(U_\alpha, V_\alpha, G_\alpha, f_\alpha : U_\alpha \to V_\alpha/G_\alpha); \ \alpha \in I,$$

satisfying the following properties:

1. *the collection $\{U_\alpha, \alpha \in I\}$ is an open covering of X;*
2. *G_α is a finite group of homeomorphisms from the open $V_\alpha \subset \mathbb{R}^n$;*
3. *$f_\alpha : U_\alpha \to V_\alpha/G_\alpha$ is a homeomorphism;*
4. *if $U_\alpha \cap U_\beta \neq \emptyset$, and if $\pi_s : V_s \to V_s/G_s$ is the natural projection (branched covering) induced by the action of G_s over V_s, then the homeomorphism*

$$f_\beta \circ f_\alpha^{-1} : f_\alpha(U_\alpha \cap U_\beta) \to f_\beta(U_\alpha \cap U_\beta)$$

can be raised to a homeomorphism

$$h_{\alpha,\beta} : \pi_\alpha^{-1}(f_\alpha(U_\alpha \cap U_\beta)) \to \pi_\beta^{-1}(f_\beta(U_\alpha \cap U_\beta)).$$

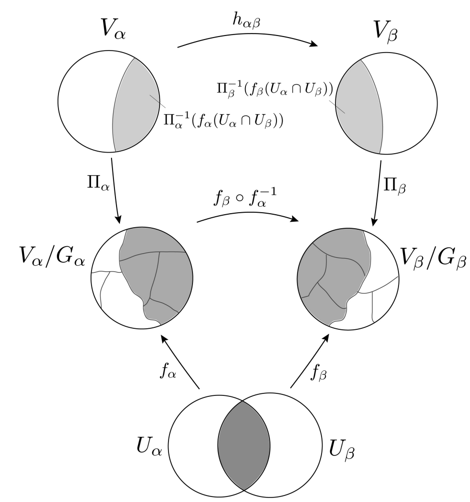

FIGURE I.6. Compatibility

Example 13. An n-dimensional real manifold M is a particular case of n-dimensional real orbifold. In this case, $G_\alpha = \{I\}$ and (U_α, f_α) are local charts of M.

Example 14. Let $X = M$ be an n-dimensional real manifold and G a group of homeomorphisms that act discontinuously in M. Then the space quotient M/G turns out to be a n-dimensional real orbifold. If the stabilizers G_x are the trivial group, then we get a n-dimensional complex manifold.

I.8. Riemann orbifold

Definition 9 (Riemann orbifold)**.** *An Riemann orbifold ϑ consists of a second countable, Hausdorff topological space X such that for each point $p \in X$ there are:*

(1) an open $U \subset X, p \in U$;
(2) an finite cyclic group G_p, generated by a conformal automorphism of the unitary disk $\mathbb{D}$;
(3) a homeomorphism $z : U \to \mathbb{D}/G$; so that if we have two of these homeomorphisms, let's say $z_1 : U_1 \to \mathbb{D}/G_1$ y $z_2 : U_2 \to \mathbb{D}/G_2$ such that $U_1 \cap U_2 \neq \emptyset$,

then

$$z_2 \circ z_1^{-1} : z_1(U_1 \cap U_2) \to z_2(U_1 \cap U_2)$$

can be raised to a holomorphic function (then biholomorphism)

In the above, $x \in X$ with $z(x) = p$ and non-trivial G_p is called a cone point of the orbifold and the order of G_p is called its cone order.

Example 15. Every Riemann surface is a Riemann orbifold. Moreover, every Riemann orbifold has a Riemann surface underlying structure.

Example 16. Let S be a Riemann surface and let G be a group of conformal automorphisms of S. If G acts discontinuously on S, then S/G is a Riemann orbifold (for instance, $S = \mathbb{H}^2$ and G a Fuchsian group). In these cases, the cone points of S/G are exactly the G-classes of those points with non-trivial G-stabilzer.

Example 17. If $G = \mathrm{PSL}(2, \mathbb{Z})$, then $\mathbb{H}^2/G$ is the Riemann orbifold whose underlying Riemann surface is the complex plane and it has two cone points, one of order 2 and the other of order 3.

CHAPTER II

Z-Orientability of Kleinian groups

II.1. On Z-orientability at the level of Kleinian groups

Let K be a discrete group of isometries of $\mathbb{X}^n$, where $\mathbb{X}^n$ is either the hyperbolic n-space $\mathbb{H}^n$ or the Euclidean n-space $\mathbb{E}^n$ or the spherical n-space $\mathbb{S}^n$, and let $P \subset \mathbb{X}^n$ be a fundamental polyhedron of K. If Γ is a proper subgroup of K, then the collection of K-translates of P induces an n-dimensional tessellation $\mathcal{T}_{K,P,\Gamma}$ on the geometric orbifold $\mathcal{O}_\Gamma = \mathbb{X}^n/\Gamma$.

Definition 10. *We say that the triple (K, P, Γ) is Z-orientable if we may label the interior of the n-faces of the tessellation $\mathcal{T}_{K,P,\Gamma}$ using only two labels such that adjacent faces have different ones.*

Remark 10. If we may label the interiors of the n-dimensional faces of tessellation with labels " + 1" or " − 1" such that adjacent faces have different labels, then we will say that (K, P, Γ) is *Z-orientable*.

Let $\mathcal{A}_P \subset K$ be the collection of side pairings of P and K_P be the subgroup of K generated by all products AB, where $A, B \in \mathcal{A}_P$. As for $A, B, C \in \mathcal{A}_P$, it holds that

$$A^2, AB, AB^{-1}, A^{-1}B, A^{-1}B^{-1}, CABC^{-1} \in K_P$$

It can be seen that K_P either coincides with K or it has index two (this happens if and only if $K_P \cap \mathcal{A}_P = \emptyset$), that is to say K_P is a normal subgroup of K. It is well known that $\mathcal{A}_P$ is a set of generators for K and that a complete set of relations is provided by how the sides of P are glued by these side-pairings (see [**1**, **6**, **15**]).

Remark 11. Let K be a group and $\mathcal{A} \subset K$ be a set of generators of K. Let $K_\mathcal{A}$ be the subgroup generated by all the elements of the form ab, where $a, b \in \mathcal{A}$.
If $Cay(K, \mathcal{A})$ is the Cayley (directed) graph of K with respect to $\mathcal{A}$, then it is a well know that $Cay(K, \mathcal{A})$ is bipartite if and only if $K_\mathcal{A}$ has a index two in K.

Let $K = \langle C_1, C_2, C_3 : C_1^r = C_2^s = C_3^t = C_1C_2C_3 = 1\rangle$, where r and s are even and t is odd. It can be see that $Cay(K, [C_1, C_2])$ is bipartite but $Cay(K, [C_2, C_3])$ is not.

Remark 12. If we chose a point p in the interior of P, then by setting $p_T = T(p)$, where $T \in K$, and joining two of these points by a simple geodesic arc if the faces containing them are adjacent (labelling such an edge by the corresponding generator in $\mathcal{A}_P$), then we obtain a representation of the Cayley graph $Cay(K, \mathcal{A}_P)$.

Lemma 1. *Let K and P be as above. Then the following are equivalent.*

(1) $(K, P, \{I\})$ *is Z-orientable.*
(2) *There is a (unique) surjective homomorphism* $\theta : K \to \mathbb{Z}_2 = \{\pm 1\}$ *such that* $\mathcal{A}_P \cap \ker(\theta) = \emptyset$.
(3) K_P *has index two in* K.

Proof. (2) $\implies$ (1). Let us label the n-face $T(P)$, $T \in K$, by $\theta(T)$. The condition $\mathcal{A}_P \cap \ker(\theta) = \emptyset$ asserts that adjacent n-faces have different labels (note that $T_1(P)$ and $T_2(P)$ are adjacent if and only if $T_2 = T_1 \circ A$, for some $A \in \mathcal{A}_P$) since $\theta(T_2) = \theta(T_1 \circ A) = \theta(T_1) \cdot \theta(A) = -\theta(T_1)$.

(2) $\implies$ (3). Let $T \in K_P$, by the definition of K_P we have that $T = AB$ where $A, B \in \mathcal{A}_P$, so $\theta(T) = \theta(AB) = \theta(A) \cdot \theta(B) = 1$ since $\mathcal{A}_P \cap \ker(\theta) = \emptyset$, so, we have to $K_P \leq \ker(\theta)$.

(3) $\implies$ (2). As K_P has index two in K, the projection map $\theta : K \to K/K_P \cong \mathbb{Z}_2$ defines a surjective homomorphism θ as in (2), and, since $\ker(\theta) = K_P$, we have $\mathcal{A}_P \cap \ker(\theta) = \emptyset$.

(1) $\implies$ (2). Just define $\theta : K \to \mathbb{Z}_2 = \{\pm 1\}$ by setting $\theta(T)$ as the label of $T(P)$, for $T \in K$, where $\theta(A) = -1$ for all $A \in \mathcal{A}_P$, in this way $\mathcal{A}_P \cap \ker(\theta) = \emptyset$. □

Next provides a necessary condition for a triple (K, P, Γ) to be Z-orientable.

Lemma 2. *If Γ is a proper subgroup of K and (K, P, Γ) is Z-orientable, then K_P has index two in K.*

Proof. We only need to ensure that we may label the interiors of all n-faces $T(P)$, with $T \in K$, using two labels and so that adjacent faces have different labels, then by Lemma 1 K_P has index two in K.

We have the hypothesis (K, P, Γ) to be *Z-orientable*, for any connected fundamental set for Γ, obtained as a union of some of the K-translates of P, we may find a labelling of these used K-translated, using the labels " + 1" and " − 1", such that adjacent ones have different labels. Below, we use this fact two times.

Let $Q \subset \mathbb{X}^n$ be a fixed connected fundamental set for Γ obtained as a union of certain K-translates of P (we may assume that P is one of them) and chose a labelling of these used K-translates of P to construct Q with labels " + 1" and " − 1" such that any two adjacent ones inside Q have different labels. Next, we translate under the action of Γ such a labelling to all n-faces $T(P)$, for $T \in K$. Note that any two n-faces in the same Γ-orbit must have the same label. Now, if there are two adjacent n-faces $T_1(P)$ and $T_2(P)$, for $T_1, T_2 \in K$, then (as

$\Gamma \neq K$), there must be another connected fundamental set Q' for the action of Γ containing them, so they must have different labels and we are done. $\square$

Lemma 3. *If K_P has index two in K and Γ is a proper subgroup of K, then (K, P, Γ) is Z-orientable if and only if $\Gamma \leq K_P$.*

Proof. As we are assuming K_P of index two in K, Lemma 1 asserts that we may fix a labelling on the n-faces $T(P)$, $T \in K$, using the labels " $+1$" and " -1" such that adjacent ones have different labels (we may assume that P has label " $+1$"). Take a sub-collection of these n-faces whose union defines a connected fundamental set Q for the action of Γ (we may assume P is contained in Q). Observe that, in order for (K, P, Γ) to be *Z-orientable*, we only need to ensure that n-faces of the same Γ-orbit must have the same label. But this last condition is equivalent to have $\Gamma \leq K_P$. $\square$

Theorem 3. *Let K be a discrete group of isometries of $\mathbb{X}^n$ and let $P \subset \mathbb{X}^n$ be a fundamental polyhedron for it. If Γ is a proper subgroup of K, then (K, P, Γ) is Z-orientable if and only if $\Gamma \leq K_P \neq K$. In particular, we may label the K-translates of P with two labels, such that adjacent ones have different ones, if and only if K_P has index two in K.*

Proof. Let K be a discrete group of isometries of $\mathbb{X}^n$ and let $P \subset \mathbb{X}^n$ be a fundamental polyhedra for it. We start with the following simple fact which states conditions for the triple $(K, P, \{I\})$ to be *Z-orientable*. The reader may note that this is just the conditions for the Cayley graph of the group K respect to the set of generators $\mathcal{A}_P$ to be bipartite (we provide a proof as matter of completeness). $\square$

Remark 13 (Grothendieck's dessins d'enfants [7])**.** Let us consider the 2-dimensional Fuchsian group $K = \Gamma(2) = \langle x(z) = z + 2, y(z) = z/(1 - z)\rangle$ (isomorphic to the free group of rank two), $P \subset \mathbb{H}^2$ a quadrilateral fundamental domain for K, whose vertices are -1, 0, 1 and ∞, and whose side pairings are given by two parabolic elements x and y. Let Γ be a finite index subgroup of K. A geodesic arc (inside P) connecting these two vertices defines a Grothendieck's dessin d'enfant $\mathcal{D}(K, P, \Gamma)$ in the compactification of $S_\Gamma = \mathbb{H}^2/\Gamma$. The 2-cell decomposition induced by $\mathcal{D}(K, P, \Gamma)$ is *Z-orientable* if and only if (K, P, Γ) is *Z-orientable*. By Belyi's theorem [2], each dessin d'enfant $\mathcal{D}(K, P, \Gamma)$ can be defined over $\overline{\mathbb{Q}}$, so the absolute Galois group acts over these combinatorial objects. In [18, 19], Zapponi applied this *Z-orientability* notion in order to produce certain unicellular dessins d'enfants in genus zero (a generalization of Leila's flowers) with the property that such a family is Galois-invariant and it contains at least two Galois orbits. In [5], *Z-orientability* was used to provide a new Galois invariant for dessins d'enfants (that is, it is not decidable from the passport, the mododromy group and the automorphism group of the dessin d'enfant).

Remark 14. Let us assume K_P has index two in K (i.e., $(K, P, \{I\})$ is *Z-orientable*). If Γ is a proper subgroup of K, then the above result asserts that if (K, P, Γ) is not *Z-orientable*, then $(K, P, \Gamma \cap K_P)$ is *Z-orientable*.

II.2. Some examples

II.2.1. Fibonacci manifolds. Let $m \geq 3$ be an integer and let us consider the fibonacci group $F(2, 2m) = \langle X_1, \ldots, X_{2m} : X_iX_{i+1} = X_{i+2},\ \ i \bmod 2m \rangle$. It is known that $F(2, 2m)$ acts as a discrete group of isometries of $\mathbb{H}^3$, with $\mathbb{H}^3/F(2, 2m)$ being a compact manifold, and that there is a fundamental polyhedron P with $\mathcal{A}_P = \{X_1, \ldots, X_{2m}\}$ [**8**, **9**]. In this case $F(2, 2m)/F(2, 2m)_P = \{I\}$, so there is no proper subgroup Γ of $F(2, 2m)$ such that $(F(2, 2m), P, \Gamma)$ is Z-orientable. In particular, the 3-faces of the tessellation of $\mathbb{H}^3$ given by the $F(2, 2m)$-translates of P cannot be labeled using two labels such that adjacent faces have different labels.

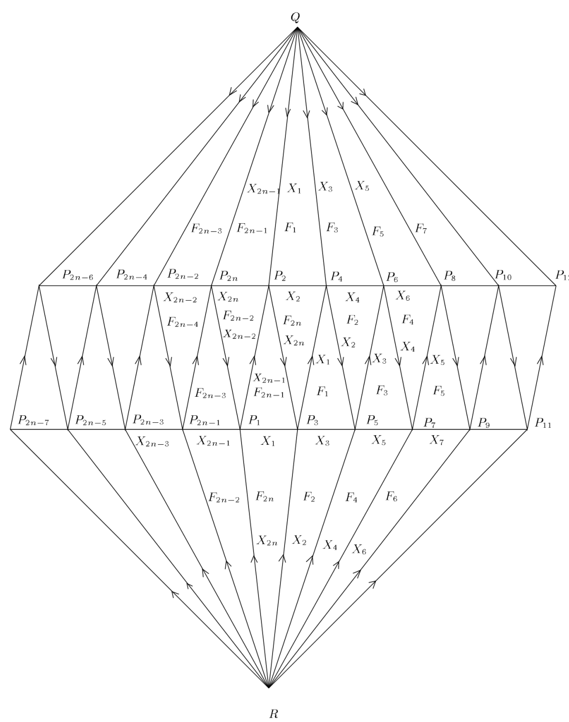

FIGURE II.1. A fundamental polyhedron for Fibonacci manifold: from the paper [**9**]

II.2.2. Reflection groups. Let us consider a convex polyhedron $P \subset \mathbb{X}^n$ and let $\mathcal{A}_P$ be the collection of the reflections on its sides. As a consequence of the Poincaré polyhedron theorem, the group K generated by the elements in $\mathcal{A}_P$ is a discrete group of isometries of $\mathbb{X}^n$ with P as a fundamental domain if all dihedral angles of P are integer parts of π. Moreover, a complete set of relations is given by $A^2 = 1$, for every $A \in \mathcal{A}_P$ and $(AB)^{n_{AB}} = 1$ for $A, B \in \mathcal{A}_P$ being the reflections on the faces Σ_A and Σ_B intersecting at an $(n-2)$-side of P with dihedral angle $\pi/n_{A,B}$ (see for instance [**1**, pp. 77]). In this case, K_P is the index two orientation-preserving half of K and, in particular, for Γ being a proper subgroup of K, the triple (K, P, Γ) is Z-orientable if and only if Γ only consists of orientation-preserving isometries.

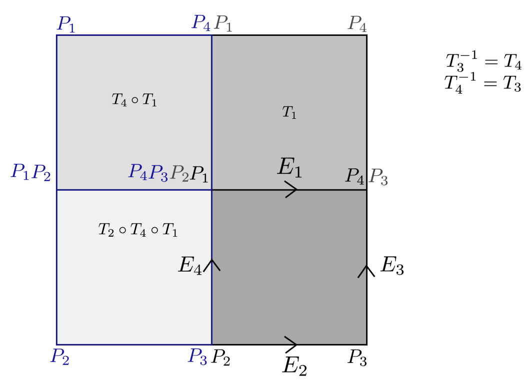

Figure II.2. A fundamental polyhedrom

II.2.3. Co-compact Fuchsian groups. Let K be a co-compact 2-dimensional Fuchsian group (that is, $\mathbb{X}^2 = \mathbb{H}^2$) with quotient orbifold $O_K = \mathbb{H}^2/K$ of genus $\gamma \geq 0$ and with $r \geq 1$ cone points. If these cone points have cone orders $2 \leq n_1, \ldots, n_r$, then we say that K has signature $(\gamma; n_1, \ldots, n_r)$ and it happens that $2\gamma - 2 + \sum_{k=1}^{r}(1 - n_k^{-1}) > 0$.

There are many different types of fundamental polygons for the group K, each one providing a "canonical" set of generators given by their side-pairings. Below we discuss only one of these situations as this will be used in the proof of Corollary 4. A tuple $\mathfrak{C} := (A_1, B_1, \dots, A_\gamma, B_\gamma, C_1, \dots, C_{r-1}) \subset K^{2\gamma+r-1}$ will be called a *generator tuple* for K if

(1) K is generated by the elements $A_1, B_1, \dots, A_\gamma, B_\gamma, C_1, \dots, C_{r-1}$; and these satisfy the relations
(2) $\left(\left(\prod_{j=1}^{\gamma}[A_j : B_j]\right)C_1\right)^{n_1} = (C_1^{-1}C_2)^{n_2} = \cdots = (C_{r-2}^{-1}C_{r-1})^{n_{r-1}} = C_{r-1}^{n_r} = 1.$

Remark 15. Let K be a Fuchsian group of signature $(\gamma; n_1, \dots, n_r)$.

(1) If $r = 1$, then the above relation (2) is just $\left(\prod_{j=1}^{\gamma}[A_j : B_j]\right)^{n_1} = 1$.
(2) (L. Keen [10]) If $\mathfrak{C} = (A_1, B_1, \dots, A_\gamma, B_\gamma, C_1, \dots, C_{r-1})$ is a generator tuple for K, then there is a fundamental polygon $P_{\mathfrak{C}}$ for K, having $4\gamma + 2(r-1)$ sides, whose side pairings set is $\mathcal{A}_{\mathfrak{C}} := \{A_1, B_1, \dots, A_\gamma, B_\gamma, C_1, \dots, C_{r-1}\}$ (see Figure II.3 and Figure III.1 for the case $\gamma = 0$).
(3) (Nielsen's theorem [11, 14]) Let $\mathfrak{C}_j := (A_{1,j}, B_{1,j}, \dots, A_{\gamma,j}, B_{\gamma,j}, C_{1,j}, \dots, C_{r-1,j})$, for $j = 1, 2$, be a generator tuple for K. Then there exists an orientation-preserving homeomorphism $F : \mathbb{H}^2 \to \mathbb{H}^2$ such that (i) $F(P_{\mathfrak{C}_1}) = P_{\mathfrak{C}_2}$, (ii) $F \circ A_{j,1} \circ F^{-1} = A_{j,2}$, $F \circ B_{j,1} \circ F^{-1} = B_{j,2}$, for $j = 1, \dots, \gamma$, and (iii) $F \circ C_{k,1} \circ F^{-1} = C_{k,2}$, for $k = 1, \dots, r-1$.

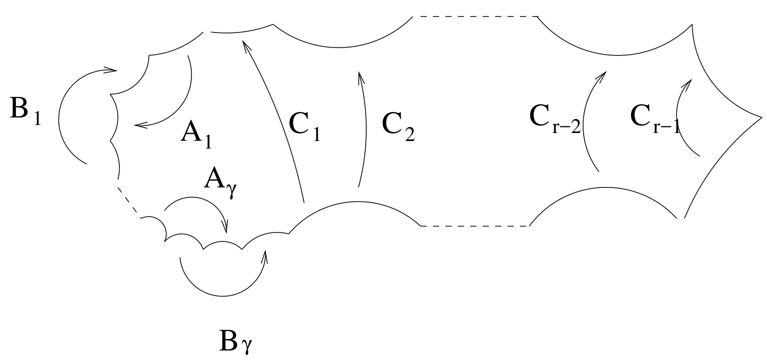

FIGURE II.3. The fundamental polygon $P_{\mathfrak{C}}$

The Z-orientability type of $(K, P_{\mathfrak{C}}, \{I\})$ can be easily checked.

Corollary 2. *Let K be a 2-dimensional Fuchsian group of signature $(\gamma; n_1, \dots, n_r)$ and let $\mathfrak{C} := (A_1, B_1, \dots, A_\gamma, B_\gamma, C_1, \dots, C_{r-1})$ be a generator tuple for it. Then $(K, P_{\mathfrak{C}}, \{I\})$ is Z-orientable if and only if n_1 and n_r are both even integers.*

Proof. By part (2) in Lemma 1, the Z-orientability of $(K, P_{\mathfrak{C}}, \{I\})$ is equivalent for the assignation $\theta(L) = -1$, for every $L \in \mathcal{A}_{\mathfrak{C}}$ to define a surjective homomorphism $\theta : K \to \{\pm 1\}$, which is equivalent to have $(-1)^{n_1} = 1 = (-1)^{n_r}$. □

Remark 16. Let K be a 2-dimensional Fuchsian group of signature $(\gamma; n_1, \dots, n_r)$ and let us consider a generator tuple $\mathfrak{C} := (A_1, B_1, \dots, A_\gamma, B_\gamma, C_1, \dots, C_{r-1})$ for K. In this case, $K_{P_{\mathfrak{C}}}$ is generated by: $C_1A_1, C_1B_1, \dots, C_1A_\gamma, C_1B_\gamma, C_1A_1^{-1}, C_1B_1^{-1}, \dots, C_1A_\gamma^{-1}, C_1B_\gamma^{-1}$, $C_1C_2, C_1C_2^{-1}, \dots, C_1C_{r-1}, C_1C_{r-1}^{-1}, C_1^2$. Corollary 2 asserts that $K_{P_{\mathfrak{C}}}$ has index two in K if and only if n_1 and n_r are both even integers, in which case, (i) a fundamental domain for it is given by the union $P_{\mathfrak{C}} \cup C_1(P_{\mathfrak{C}})$, the above generators are the corresponding side-pairings (see Figure III.2) and (ii) $\mathbb{H}^2/K_{P_{\mathfrak{C}}}$ has signature $(2\gamma; n_1/2, n_2, n_2, \dots, n_{r-1}, n_{r-1}, n_r/2)$ and it admits a conformal automorphism of order two, induced by C_1, with two fixed points (these being the two cone points of order $n_1/2$ and $n_r/2$) and permuting the other $2(r-2)$ cone points.

Corollary 3. *Let K be a 2-dimensional Fuchsian group of signature $(\gamma; n_1, \dots, n_r)$ with a generator tuple $\mathfrak{C} := (A_1, B_1, \dots, A_\gamma, B_\gamma, C_1, \dots, C_{r-1})$. Let us assume that n_1 and n_r are both even integers. If Γ is a proper subgroup of K, then $(K, P_{\mathfrak{C}}, \Gamma)$ is Z-orientable if and only if Γ is a subgroup of the index two subgroup of K generated by the elements $C_1A_j, C_1B_j, C_1A_j^{-1}, C_1B_j^{-1}, C_1C_k, C_1C_k^{-1}$ and C_1^2, where $j = 1, \dots, \gamma$, and $k = 2, \dots, r-1$.*

Corollary 4. *Let K be a 2-dimensional Fuchsian group of signature $(0; n_1, n_2, n_3)$ and with generator tuple $\mathfrak{C} := (C_1, C_2)$, that is, $K = \langle C_1, C_2 : C_1^{n_1} = (C_1^{-1}C_2)^{n_2} = C_2^{n_3} = 1\rangle$. If Γ is a proper finite index subgroup of K, then $(K, P_{\mathfrak{C}}, \Gamma)$ is Z-orientable if and only if (i) n_1 and n_3 are even integers and (ii) $\Gamma \leq \langle C_1C_2, C_1C_2^{-1}, C_1^2\rangle$.*

CHAPTER III

Application: Square roots of meromorphic maps

III.1. On square roots of meromorphic maps

Let S be a connected (not necessarily compact) Riemann surface and let $\varphi : S \to \widehat{\mathbb{C}}$ be a surjective meromorphic map. We assume φ to be of finite type, that is, (i) its set $B(\varphi)$ of branch values is finite and (ii) for every $p \in B(\varphi)$ the set $M_p \subset \{1, 2, \ldots\}$ of local degrees of φ at the preimages of p is bounded. In the case that $S \cong \widehat{\mathbb{C}}$ we also assume that the degree of φ is at least two.

Under the assumption (i), the set $B(\varphi)$ of branch values of φ has finite cardinality at least two, and we set $B(\varphi) = \{p_1, \ldots, p_n\}$. Under condition (ii), we may define the branch order of each $p \in B(\varphi)$ as the least common multiple of all the values in M_p.

A simple arc $\delta \subset \widehat{\mathbb{C}}$, with end points two different branch values $p_i, p_j \in B(\varphi)$ and containing all the branch values, will be called an *admissible arc for* (S, φ) *with respect to* $\{p_i, p_j\}$. In this case, $\varphi^{-1}(\delta)$ induces a 2-cell decomposition $\mathcal{F}_{\varphi,\delta}$ on S. Set $\varphi_{i,j} = (\varphi - p_i)/(\varphi - p_j)$ (where the factor $(\varphi - p_j)$ is deleted in the case that $p_j = \infty$).

In the following result we observe that Z-orientability property of $\mathcal{F}_{\varphi,\delta}$ provides the following simple geometrical statement for the existence of square roots of $\varphi_{i,j}$, that is, a meromorphic surjective map $\psi : S \to \widehat{\mathbb{C}}$ satisfying $\varphi_{i,j} = \psi^2$.

Theorem 4. *Let S be a connected Riemann surface and $\varphi : S \to \widehat{\mathbb{C}}$ be a surjective meromorphic map of finite type and let $p_i, p_j \in B(\varphi)$, $p_i \neq p_j$. If δ_1 and δ_2 are any two admissible arcs for (S, φ) with respect to $\{p_i, p_j\}$, then $\mathcal{F}_{\varphi,\delta_1}$ is Z-orientable if and only if $\mathcal{F}_{\varphi,\delta_2}$ is Z-orientable, and this is equivalent to the existence of an square root of $\varphi_{i,j}$.*

Corollary 5. *Let S be a connected Riemann surface and $\varphi : S \to \widehat{\mathbb{C}}$ be a surjective meromorphic map of finite type such that* $0, \infty \in B(\varphi)$*. If δ is an admissible arc for* (S, φ) *with respect to* $\{0, \infty\}$*, then $\mathcal{F}_{\varphi,\delta}$ is Z-orientable if and only if there is a meromorphic map $\psi : S \to \widehat{\mathbb{C}}$ such that $\varphi = \psi^2$.*

Let S be a connected Riemann surface and $\varphi : S \to \widehat{\mathbb{C}}$ be a surjective meromorphic map (with at least two branch values and of finite type). As for any rational map R of degree two there are Möbius transformations $L, T \in \mathrm{PSL}_2(\mathbb{C})$ such that $T \circ R \circ L(z) = z^2$, we may rewrite Theorem 4 as follows.

Corollary 6. *Let S be a connected Riemann surface and $\varphi : S \to \widehat{\mathbb{C}}$ be a surjective meromorphic map (with at least two branch values) and of finite type. Let δ be a simple arc passing through all branch values and whose end points are also branch values. Then the 2-cell $\mathcal{F}_{\varphi,\delta}$ in S is Z-orientable if and only if there is a rational map R of degree two and there is a meromorphic map $\psi : S \to \widehat{\mathbb{C}}$ such that $\varphi = R(\psi)$.*

Remark 17. Let S be a connected Riemann surface and $\varphi : S \to \widehat{\mathbb{C}}$ a surjective meromorphic map of finite type, with at leat two branch values (that is, for $S \cong \widehat{\mathbb{C}}$ we assume the degree of φ at least two). As the property to have an square root is a Galois invariant, the above asserts that the Z-orientability property of a pair (S, φ) (with respect to a pair of its branch values) as above is a $\mathrm{Gal}(\mathbb{C})$-invariant. . Let us assume $B(\varphi) = \{p_1, \ldots, p_n\}$ and, as before we consider the $n(n-1)/2$ meromorphic maps $\varphi_{i,j} = (\varphi - p_i)/(\varphi - p_j)$, for $1 \leq i < j \leq n$. One may define the value $\mathrm{TOT}(S, \varphi)$ as the number of those ones having an square root. This number can be easily obtained by checking the Z-orientability of the 2-cells $\mathcal{F}_{\varphi,\delta_{i,j}}$, where $\delta_{i,j}$ is a fixed admissible arc with respect to $\{p_i, p_j\}$. For $n \geq 3$ odd, one may check that the product $\prod_{1 \leq i < j \leq n} \varphi_{i,j}$ is an square, so $\mathrm{TOT}(S, \varphi) \in \{0, 1, \ldots, n(n-1)/2 - 2, n(n-1)/2\}$. For $n = 3$, it was observed in [5] that TOT is a new Galois invariant for dessins d'enfants.

III.2. Proof of theorem 4

Let S be a connected Riemann surface and $\varphi : S \to \widehat{\mathbb{C}}$ be a surjective meromorphic map of finite type whose set of branch values contains the points p_i and p_j, where $p_i \neq p_j$. Let $T(z) = (z - p_i)/(z - p_j)$ (where for $p_k = \infty$ we delete the factor $(z - p_k)$). Then the meromorphic map $\varphi_{i,j} = T(\varphi)$ has 0 and ∞ as branch values. Without lost of generality, we may then assume $p_i = 0$ and $p_j = \infty$, that is, $\varphi = \varphi_{i,j}$.

Let $r \geq 2$ be the cardinality of the set of the branch values of φ (necessarily, $r \geq 3$ if S is not isomorphic to $\widehat{\mathbb{C}}$). If $p \in \widehat{\mathbb{C}}$ is a branch value, then its branch order is given as the minimum common multiple of the local degrees of φ at all the points in $\varphi^{-1}(p)$ (here we are using the condition on the upper bound on the local degrees of φ at the preimages of its brach values). Let $\delta \subset \widehat{\mathbb{C}}$ an admissible arc for (S, φ), that is, a simple arc through all branch values of φ, starting at the branch value $p_1 = 0$, ending at the branch value $p_r = \infty$. We label the rest of the branch values of φ as $p_2, \ldots, p_{r-1}$, such that the branch value p_j is between the branch values p_{j-1} and p_{j+1}. Let us denote by n_j be the branch order of p_j. We set $\mathbb{X}^2$ to be either $\widehat{\mathbb{C}}$ or $\mathbb{C}$ or $\mathbb{H}^2$ depending if $\sum_{k=1}^{r}(1 - n_k^{-1}) - 2$ is negative, zero or positive, respectively. Let K be a discrete group of isometries of $\mathbb{X}^2$ such that $\mathbb{X}^2/K$ is the orbifold of genus zero whose cone points are the branch values of φ with cone orders being the corresponding branch orders. So there is a proper subgroup Γ of K so that $\varphi : S \to \widehat{\mathbb{C}}$

is induced by the inclusion $\Gamma < K$ (here we are using the fact that φ is not one-to-one and surjective). The arc δ defines a fundamental domain P_δ (see Figure III.1), with $2(r-1)$ sides, and side pairings $C_1, \ldots, C_{r-1}$, so that

$$K = \langle C_1, \ldots, C_{r-1} : C_1^{n_1} = (C_1^{-1}C_2)^{n_2} = \cdots = (C_{r-2}^{-1}C_{r-1})^{n_{r-1}} = C_{r-1}^{n_r} = 1\rangle,$$

in other words, $(C_1, \ldots, C_{r-1})$ is a generator tuple for K. (The image of the K-translates of such a fundamental polygon induces the 2-cell decomposition $\mathcal{F}_{\varphi,\delta}$ on $S = \mathbb{X}^2/\Gamma$).

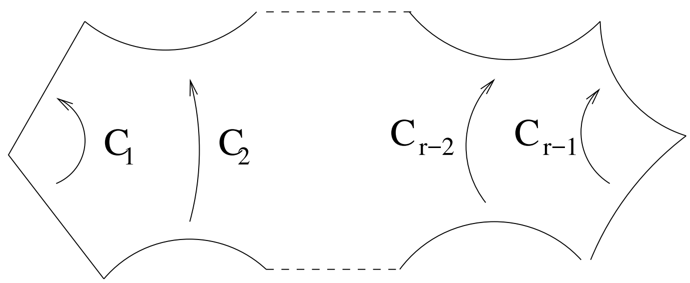

FIGURE III.1. The fundamental polygon P_δ

By Corollary 2, $K_{P_\delta} = \langle C_1^2, C_1C_2, C_1C_2^{-1}, \ldots, C_1C_{r-1}, C_1C_{r-1}^{-1}\rangle$ has index two in K if and only if n_1 and n_r are even integers. Note that this condition is independent on the choice of the admissible arc δ (it only depends on the branch orders of the branch values 0 and ∞).

If K_{P_δ} has index two in K, then the orbifold $\mathbb{X}^2/K_{P_\delta}$ is the Riemann sphere with exactly $2(r-1)$ cone points, these being of orders $n_1/2, n_2, n_2, \ldots, n_{r-1}, n_{r-1}, n_r/2$ (see Figure III.2 for a fundamental domain for K_{P_δ}).

This provides a degree two meromorphic map $\eta : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ branched at the end points of δ, ie., 0 and ∞,and $\eta^{-1}(\delta)$ is a loop (containing all the cone points of $\mathbb{H}^2/K_{P_\delta}$). We may in fact assume that $\eta(z) = z^2$. An inclusion $\Gamma \leq K_{P_\delta} \leq K$ ensures the existence of a meromorphic map $\psi : S \to \widehat{\mathbb{C}}$ such that $\varphi = \eta \circ \psi = \psi^2$. The other direction is clear, if there is a meromorphic map $\psi : S \to \widehat{\mathbb{C}}$ so that $\varphi = \psi^2$, then K_{P_δ} has index two in K and $\Gamma \leq K_{P_\delta}$. All the above asserts that the existence of an square root of φ is equivalent to have $\Gamma \leq K_{P_\delta}$ and both integers n_1 and n_r are even. By Corollary 3, this is equivalent for (K, P_δ, Γ) to be Z-orientable, that is, the faces of the 2-cell decomposition on S induced by the liftings under φ of δ can be labelled with two labels such that adjacent faces have different labels. The last part of the theorem follows from the fact that for φ to have an square root does not depends on the choice of the admissible arc (see also Remark 18).

Remark 18. Given the branch values p_i and p_{i+1} of φ, $i = 1, \ldots, r-2$, we may consider a Dehn-twist D_i; an orientation-preserving diffeomorphism of $\widehat{\mathbb{C}}$ which permutes p_i with p_{i+1} and fixes each of the other branch values. The arc $D_i(\delta)$ is a simple arc with end points 0

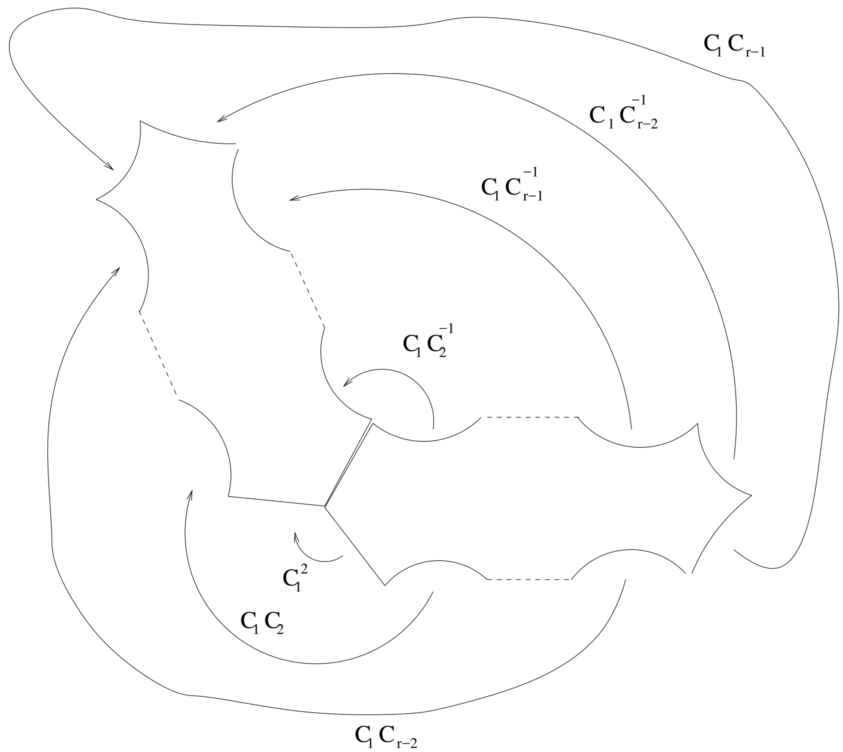

FIGURE III.2. A fundamental domain for K_{P_δ} in the case $K/K_{P_\delta} \cong \mathbb{Z}_2$

and ∞, passing through all branch values of φ, and in the following order

$$(0 = p_1, \dots, p_{i-1}, p_{i+1}, p_i, p_{i+2}, \dots, p_r = \infty).$$

The above diffeomorphism induces a geometrical automorphism Θ of K defined by the rule $(C_1, \dots, C_r) \mapsto (C_1, \dots, C_{i-1}, C_{i+1}, C_{i+1}^{-1} C_i C_{i+1}, C_{i+2}, \dots, C_r)$. The group $K_{P_{D_i(\delta)}}$ is the image of K_{P_δ} under the above automorphism. As $\Theta(C_1^2) = C_1^2$, $\Theta(C_1C_2) = C_1C_2$, $\Theta(C_1C_2^{-1}) = C_1C_2^{-1}, \dots, \Theta(C_1C_{i-1}) = C_1C_{i-1}$, $\Theta(C_1C_{i-1}^{-1}) = C_1C_{i-1}^{-1}$, $\Theta(C_1C_{i+2}) = C_1C_{i+2}$, $\Theta(C_1C_{i+2}^{-1}) = C_1C_{i+2}^{-1}, \dots, \Theta(C_1C_{r-1}) = C_1C_{r-1}$, $\Theta(C_1C_{r-1}^{-1}) = C_1C_{r-1}^{-1}$, $\Theta(C_1C_i) = C_1C_{i+1}$, $\Theta(C_1C_i^{-1}) = C_1C_{i+1}^{-1}$, $\Theta(C_1C_{i+1}) = C_1C_{i+1}^{-1}C_iC_{i+1} = (C_1C_{i+1}^{-1})(C_1^{-2}C_1C_i^{-1})^{-1}C_1^{-2}(C_1C_{i+1})$, $\Theta(C_1C_{i+1}^{-1}) = C_1C_{i+1}^{-1}C_i^{-1}C_{i+1} = (C_1C_{i+1}^{-1})(C_1^{-2}C_1C_i)^{-1}C_1^{-2}(C_1C_{i+1})$, we may observe that $K_{P_{D_i(\delta)}} = K_{P_\delta}$.

Remark 19. Let us assume (S, φ) as above with S a closed Riemann surface of genus $g \geq 1$, such that 1 is not a branch value of it. Let δ be an admissible arc for (S, φ) with $1 \notin \delta$. Let $\mathcal{G}^v_{\varphi,\delta}$ be the dual graph associated to the 2-cell decomposition $\mathcal{F}_{\varphi,\delta}$ of S (its vertices are center of faces of $\mathcal{F}_{\varphi,\delta}$ and two such vertices are joined by an edge if the corresponding faces are adjacent). Assume that $\mathcal{F}_{\varphi,\delta}$ is Z-orientable. This is equivalent for $\mathcal{G}^v_{\varphi,\delta}$ to be bipartite, so it defines a dessin d'enfant on S. Corollary 5 asserts the existence of a meromorphic map $\psi : S \to \widehat{\mathbb{C}}$ with $\varphi = \psi^2$. Then $(\psi+1)^2/4\psi$ is a Belyi map on S whose dessin d'enfant has a bipartite graph homotopic to $\mathcal{G}^v_{\varphi,\delta}$.

Remark 20. Let (S,φ) be a Belyi pair and let $\mathcal{D}_\varphi$ be the dessin d'entant $\mathcal{D}_\varphi$ induced by the bipartite graph $\mathcal{G}_\varphi = \varphi^{-1}([0,1])$. In this case, $\mathcal{D}_\varphi$ is Z-orientable if and only if for an admissible arc δ for $(S,\varphi/(\varphi-1))$ the corresponding 2-cell decomposition $\mathcal{F}_{\varphi/(\varphi-1),\delta}$ is Z-orientable. In particular, Corollary 5 implies that $\mathcal{D}_\varphi$ is Z-orientable if and only if $\varphi/(\varphi-1)$ has an square root [**5**].

III.3. An example

Let $\tau \in \mathbb{H}^2$ and its corresponding Weierstrass $\wp$-function

$$\wp(z:\tau) = z^{-2} + \sum_{\omega\in\Lambda_\tau-\{0\}} \left((z-\omega)^{-2} - \omega^{-2}\right),$$

where $\Lambda_\tau = \mathbb{Z}\oplus\tau\mathbb{Z}$. If we set $e_1(\tau) = \wp(1/2;\tau)$, $e_2(\tau) = \wp((1+\tau)/2;\tau)$ and $e_3(\tau) = \wp(\tau/2;\tau)$, then $\varphi_\tau : \mathbb{C} \to \widehat{\mathbb{C}}$, defined as

$$\varphi_\tau(z) = \frac{\wp(z;\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)},$$

is a surjective meromorphic map whose branch values are 0, 1, ∞ and $\lambda(\tau) = \varphi_\tau(e_3(\tau))$, each one of order two. In this case, φ_τ is a regular branched covering with deck group $K = \langle A(z) = z+1, U(z) = 1-z, V(z) = \tau+1-z\rangle$. A fundamental polygon P for K is a 4-side polygon with vertices at 0, 1, $\tau/2+1$ and $\tau/2$, with the above generators as side-pairings. In this case $K_P = \langle A^2, AV, AU\rangle$, which is of index two. It follows, from Theorem 4, that $\varphi_\tau = \psi_\tau^2$ for a suitable meromorphic map $\psi_\tau : \mathbb{C} \to \widehat{\mathbb{C}}$. The branch values of ψ_τ are given by the points ± 1 and $\pm\sqrt{\lambda(\tau)}$, all of them with order two. This corresponds to the following facts. Let $G = \langle A, B = VU\rangle \cong \mathbb{Z}^2$, $\mathbb{Z}^2 \cong L = \langle A^2$ and $B\rangle \leq K_P$. There are regular holomorphic covering maps $\pi_L : \mathbb{C} \to T_L$ (with deck group L), $\pi_G : \mathbb{C} \to T_G$ (with deck group G), where T_L and T_G are genus one Riemann surfaces, there is a two-fold cover map $\pi : T_L \to T_G$, a branched two-fold cover $\rho : T_G \to \widehat{\mathbb{C}}$ (branched at ∞, 0, 1 and $\lambda(\tau)$) and a branched two-fold cover $\eta : T_L \to \widehat{\mathbb{C}}$ (branched at ± 1 and $\pm\sqrt{\lambda(\tau)}$) such that $\eta\circ\pi_L = \psi_\tau$, $\rho\circ\pi_G = \varphi_\tau, \eta^2 = \rho\circ\pi$. The surface T_L corresponds to the elliptic curve $y^2 = (x^2-1)(x^2-\lambda(\tau))$, the surface T_G corresponds to $v^2 = u(u-1)(u-\lambda(\tau))$ and $\pi(x,y) = (x^2, xy)$.

Remark 21. Let S be a connected Riemann surface and let $\varphi : S \to \widehat{\mathbb{C}}$ be a surjective meromorphic map of finite type. Let $\{p_1 = 0, p_2, \ldots, p_{r-1}, p_r = \infty\}$ be its set of branch values and set the quadratic meromorphic form $\omega = d\varphi^2/\varphi\prod_{j=2}^{r-1}(\varphi - p_j)^2$. By Corollary 4, ω is the square of a meromorphic 1-form if and only if the 2-cell decomposition $\mathcal{F}_{\varphi,\delta}$ is Z-orientable. The following facts can be directly observed. (1) If $z \in \varphi^{-1}(0)$ is a zero of order $N \geq 1$, then z is (a) a simple pole of ω if $N = 1$; (b) a zero of order $N-2$ of ω if

$N \geq 3$; and (c) a non-zero regular point of ω if $N = 2$. (2) If $z \in \varphi^{-1}(\infty)$ is a pole of order $N \geq 1$, then z is (a) a zero of order $(2r-5)N-2$ of ω if $r \geq 4$; (b) a zero of order $N-2$ of ω if $r = 3$ and $N \geq 3$; (c) a simple pole of ω if $r = 3$ and $N = 1$; and (d) a non-zero regular point if $r = 3$ and $N = 2$. (3) If $z \in \varphi^{-1}(p_2) \cup \cdots \cup \varphi^{-1}(p_{r-1})$, then z is a pole of order two of ω. The above asserts that, for ω to be a Strebel quadratic meromorphic form, necessarily every zero of φ must have order at least two and, for $r = 3$ every of its poles must have order at least two. We may draw $r-2$ simple arcs connecting 0 with ∞, all of them decomposing $\widehat{\mathbb{C}}$ into $r-2$ discs, each one containing exactly of the points p_j, $j = 2, \ldots, r-1$. The graph $\mathcal{H}$ obtained by lifting these arcs under φ corresponds (up to isotopy) to the non-compact horizontal lines of ω. If $r = 3$ and $p_2 = 1$ (so (S, φ) is a Belyi pair), then the map $\beta = \varphi/(\varphi - 1) : S \to \widehat{\mathbb{C}}$ is a Belyi map whose associated dessin d'enfant is defined by the bipartite graph $\mathcal{H}$ (vertices in the φ-preimage of 0 are the white ones and those of ∞ the black ones) and $\omega = d\beta^2/\beta(\beta - 1)$. This case was studied by Zapponi in [**17**, **19**] and also by Mulase-Penkava in [**13**].

CHAPTER IV

Generalization

IV.1. On n-square root of meromorphic maps

Let S be a connected (not necessarily compact) Riemann surface and let $\varphi : S \to \widehat{\mathbb{C}}$ be a surjective meromorphic map of finite type.

If $p \in B_\varphi$ and $q \in \varphi^{-1}(p)$, then we denote by $m_\varphi(q)$ the local degree of φ at q. As φ is of finite type, the set $M_p = \{m_\varphi(q) : q \in \varphi^{-1}(p)\}$ is bounded. The branch order of $p \in B_\varphi$ as the minimum common multiple of M_p.

If $n \geq 2$ is an integer, then a meromorphic map $\psi : S \to \widehat{\mathbb{C}}$ is called n-square root of φ if $\varphi(q) = (\psi(q))^n$, for every $q \in S$. Note that necessarily for this to happen one must have that $\infty, 0$ are branch values of φ and that their branch orders are both divisible by n.

Let us observe that, in order for the existence of some n-square root of φ, necessarily the branch orders at 0 and ∞ must be both divisible by n.

Let us order the branch values, $B_\varphi = \{p_1 = 0, p_2, \ldots, p_{r-1}, p_r = \infty\}$. We have associated the tuple $(k_1, \ldots, k_r)$, where k_j is the branch order of φ at the branch value p_j, called the *signature* of φ respect to the given ordering.

An *admisible arc* for φ, with respect to the given ordering, is a simple arc $\delta \subset \widehat{\mathbb{C}}$ whose end points are 0 and ∞, $B_\varphi \setminus \{0, \infty\}$ is contained in the interior of δ and p_j is between p_{j-1} and p_{j+1}. If we are not interested in the ordering of the interior points, we just talk of an admissible arc for φ. For the admissible arc δ the graph $\varphi^{-1}(\delta) = \mathcal{G}_\delta \subset S$ defines a map $\mathcal{F}_\delta$ on S, where each of it faces is a polygon with $2(r-1)$ sides.

We say that $\mathcal{F}_\delta$ is *n-Z-orientable* if we may label its faces using the labels $1, 2, \ldots, n$, such that the following properties hold:

(1) around each vertice $q \in \varphi^{-1}(0)$, following the counterclockwise orientation, the labelling is a finite sequence of $(1, 2, \ldots, n)$.
(2) around each vertice $q \in \varphi^{-1}(\infty)$, following the clockwise orientation, the labelling is a finite sequence of $(1, 2, \ldots, n)$.
(3) around each vertice $q \notin \varphi^{-1}\{0, \infty\}$, we see alternately a finite sequence of only two consecutive labels $(i, i+1)$, if $i = 1, \ldots, n-1$, or $(1, n)$.

Remark 22. The 2-Z-orientable definition coincides with the one provided by Zapponi in [**17**, **18**, **19**]. This notion (where he used the term "orientable") was used by Zapponi to decide if a given Strebel quadratic meromorphic form [**16**] on a closed Riemann surface has an square root (in this case the 2-cell decomposition is the one defined by the graph whose vertices are the zeroes of the form and the edges are the non-compact horizontal trajectories). The n-Z-orientable property is a particular case of a θ-Zapponi-orientability for general Kleinian groups as described in Section IV.2.

Next extends the results in [**4**] provided for the case $n = 2$.

Theorem 5. *Let $n \geq 2$ be an integer, S be a connected Riemann surface and $\varphi : S \to \widehat{\mathbb{C}}$ be a surjective meromorphic map of finite type such that (i) $0, \infty \in B_\varphi$ and (ii) the branch order of 0 and ∞ are both divisible by n. If δ is an admissible arc for φ, then the existence of an n-square root of φ is equivalent for $\mathcal{F}_\delta$ to be n-Z-orientable.*

Observe that the above result permits to see the following fact.

Corollary 7. *Let $n \geq 2$ be an integer, S be a connected Riemann surface and $\varphi : S \to \widehat{\mathbb{C}}$ be a surjective meromorphic map of finite type such that (i) $0, \infty \in B_\varphi$ and (ii) the branch order of 0 and ∞ are both divisible by n. If δ_1 and δ_2 are admissible arcs for φ, then $\mathcal{F}_{\delta_1}$ is n-Z-orientable if and only if $\mathcal{F}_{\delta_2}$ is n-Z-orientable.*

Remark 23. Let S be a closed Riemann surface and $\varphi : S \to \widehat{\mathbb{C}}$ a surjective meromorphic map with $0, \infty \in B_\varphi$. As the property to have an n-square root is a Gal($\mathbb{C}$)-invariant, the above asserts that the n-Z-orientability property of a pair (S, φ) is a Gal($\mathbb{C}$)-invariant.

IV.2. θ-Zapponi-orientability of Kleinian groups

Let K be a discrete group of isometries of $\mathbb{X}^m$, where $\mathbb{X}^m$ is either the hyperbolic m-space $\mathbb{H}^m$ or the Euclidian n-space $\mathbb{E}^m$ or the spherical m-space $\mathbb{S}^m$. Let $P \subset \mathbb{X}^m$ be a fundamental polyhedron of K and let $\mathcal{A}_P$ the subset of K consisting of the side-pairings of P. It is well known that $\mathcal{A}_P$ is a set of generators for K and that a complete set of relations is provided by how the sides of P are glued by these side-pairings (see [**1**, **6**, **15**]). The K-translates of P provides a n-tessellation $\mathcal{T}_{K,P}$ of $\mathbb{X}^m$.

Assume we are given a surjective homomorphism $\theta : K \to G$, where G is a finite group. Such a homomorphism permits to label the faces of $\mathcal{T}_{K,P}$; the face $T(P)$ is labelled with the element $\theta(T) \in G$. It can be seen that adjacent faces have different labels if and only if $\mathcal{A}_P \cap \ker(\theta) = \emptyset$. If this is the situation, then we say that θ is (K, P)-*admissible*.

Example 18 ($G = \mathbb{Z}_2$)**.** If K_P is the subgroup of K generated by all the elements of the form AB, where $A, B \in \mathcal{A}_P$, then either $K_P = K$ or its has index two in K. If $\theta : K \to G = \mathbb{Z}_2$ is any homomorphism, then $K_P \leq \ker(\theta)$. It follows that θ is (K, P)-admissible if and only if $K \neq K_P = \ker(\theta)$ (in particular, there is at most one (K, P)-admissible homomorphism onto $\mathbb{Z}_2$).

Example 19 ($G = \mathbb{Z}_n$)**.** Let $n, r \geq 2$ and let us consider a Fuchsian group, acting in the hyperbolic plane $\mathbb{H}^2$, with the following presentation

$$K = \langle C_1, \dots, C_{r-1} : C_1^{k_1} = (C_1^{-1}C_2)^{k_2} = \cdots = (C_{r-2}^{-1}C_{r-1})^{k_{r-1}} = C_{r-1}^{k_r} = 1\rangle.$$

We may consider the fundamental domain P as shown in Figure IV.1, whose set of side-pairings is $\mathcal{A}_P = \{C_1, \dots, C_{r-1}\}$. Let $G = \langle \sigma = (1, \dots, n)\rangle \cong \mathbb{Z}_n$. If k_1 and k_r are both multiples of n, then we may consider the surjective homomorphism $\theta_0 : K \to G$, defined by $\theta_0(C_j) = \sigma$, for every $j = 1, \dots, r-1$. As $\ker(\theta_0)$ is the group generated by the conjugates of the elements

$$C_1^n, C_1^{-1}C_2, C_1C_2^{-1}, \dots, C_1^{-1}C_{r-1}, C_1C_{r-1}^{-1},$$

it follows that θ_0 is (K, P)-admissible. The induced labelling of $\mathcal{T}_{K,P}$ by θ_0 satisfies to be n-Z-orientable.

Remark 24. If K is a discrete group of isometries of $\mathbb{X}^m$ and $\theta : K \to G$ is a surjective homomorphism to a finite group G, then it is possible to find a fundamental polyhedron P for K such that θ is (K, P)-admissible.

Let us assume that θ is (K, P)-admissible and Γ is a proper subgroup of K. The tessellation $\mathcal{T}_{K,P}$ induces an m-dimensional tessellation $\mathcal{T}_{K,P,\Gamma}$ on the geometric orbifold $\mathcal{O}_\Gamma = \mathbb{X}^n/\Gamma$. The labelling on the faces of $\mathcal{T}_{K,P}$, provided by the (K, P)-admissible homomorphism θ, induces a labelling of the faces of the tessellation $\mathcal{T}_{K,P,\Gamma}$. It is not difficult to see that the adjacent faces of this last tessellation have different labels if and only if $\Gamma \leq \ker(\theta)$. If this is the situation, we say that (K, P, Γ) is *θ-Z-orientable*.

Summarizing all the above is the following.

Proposition 1. *Let K be a discrete group of isometries of $\mathbb{X}^m$, $P \subset \mathbb{X}^m$ be a fundamental polyhedron for it and $\mathcal{A}_P \subset K$ be the set of side-pairings of P. Let $\theta : K \to G$ be some homomorphism onto a finite group G. Then:*

(1) θ is (K, P)-admissible if and only if $\mathcal{A}_P \cap \ker(\theta) = \emptyset$.
(2) If $\theta : K \to G$ is an admissible homomorphism and Γ is a proper subgroup of K, then (K, P, Γ) is θ-Z-orientable if and only if $\Gamma \leq \ker(\theta)$.

IV.3. Proof of theorem 5

Let $r \geq 2$ be the cardinality of B_φ and let $\delta \subset \widehat{\mathbb{C}}$ an admissible arc for φ, that is, a simple arc through all branch values of φ, starting at the branch value $p_1 = 0$, ending at the branch value $p_r = \infty$. We label the rest of the branch values of φ as $p_2, \dots, p_{r-1}$, such that the branch value p_j is between the branch values p_{j-1} and p_{j+1}. Let us denote by k_j be the branch order of p_j. We know that k_1 and k_r are both multiple of n.

We set $\mathbb{X}^2$ to be either $\widehat{\mathbb{C}}$ or $\mathbb{C}$ or $\mathbb{H}^2$ depending if $\sum_{j=1}^{r}(1-k_j^{-1})-2$ is negative, zero or positive, respectively. Let K be a discrete group of isometries of $\mathbb{X}^2$ such that $\mathbb{X}^2/K$ is the orbifold of genus zero whose cone points are the branch values of φ with cone orders being the corresponding branch orders. So there is a proper subgroup Γ of K so that $\varphi : S \to \widehat{\mathbb{C}}$ is induced by the inclusion $\Gamma < K$ (here we are using the fact that φ is not one-to-one and surjective). The arc δ defines a fundamental domain P_δ (see Figure IV.1), with $2(r-1)$ sides, and side pairings $C_1, \ldots, C_{r-1}$, so that

$$K = \langle C_1, \ldots, C_{r-1} : C_1^{k_1} = (C_1^{-1}C_2)^{k_2} = \cdots = (C_{r-2}^{-1}C_{r-1})^{k_{r-1}} = C_{r-1}^{k_r} = 1\rangle,$$

in other words, $(C_1, \ldots, C_{r-1})$ is a generator tuple for K. (The image of the K-translates of such a fundamental polygon induces the n-cell decomposition $\mathcal{F}_\delta$ on $S = \mathbb{X}^2/\Gamma$).

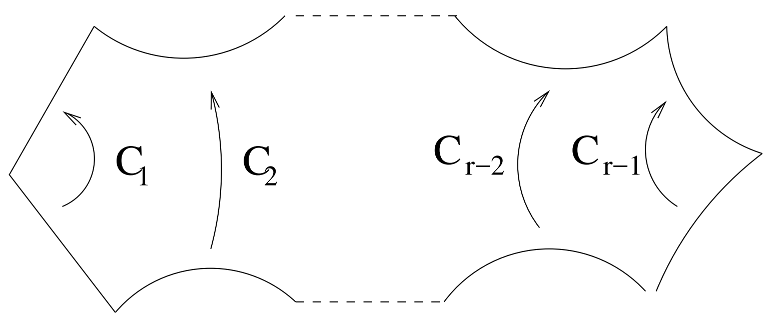

FIGURE IV.1. The fundamental polygon P_δ

As k_1 and k_r are multiples of n, we may consider the surjective homomorphism θ_0 as defined in Example 19. The orbifold $\mathbb{X}^2/\ker(\theta_0)$ is the Riemann sphere with exactly $2(r-1)$ cone points, these being of orders $k_1/n, k_2, k_2, \ldots, k_{r-1}, k_{r-1}, k_r/n$.

The pair $(K, \ker(\theta_0))$ induces a degree n meromorphic map $\eta : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ branched at the end points of δ, ie., 0 and ∞,and $\eta^{-1}(\delta)$ is a loop (containing all the cone points of $\mathbb{H}^2/\ker(\theta_0)$). We may assume $\eta(z) = z^n$.

Lemma 4. $\Gamma \leq \ker(\theta_0)$ *if and only if there exists a meromorphic map* $\psi : S \to \widehat{\mathbb{C}}$ *such that* $\varphi = \eta \circ \psi = \psi^n$.

PROOF. An inclusion $\Gamma \leq \ker(\theta_0) \leq K$ ensures the existence of a meromorphic map $\psi : S \to \widehat{\mathbb{C}}$ such that $\varphi = \eta \circ \psi = \psi^n$. Conversely, if there is a meromorphic map $\psi : S \to \widehat{\mathbb{C}}$ so that $\varphi = \psi^n$, then $\Gamma \leq \ker(\theta_0)$. $\square$

The above, together the following lemma (and the fact that for φ to have an n-square root does not depends on the choice of the admissible arc) ends the proof.

Lemma 5. $\Gamma \leq \ker(\theta_0)$ *if and only if* $\mathcal{F}_\delta$ *is n-Z-orientable.*

Proof. If $\Gamma \leq \ker(\theta_0)$, then by the previous lemma, we may assume that $\varphi = \psi^n$. Then the map on the Riemann sphere obtained by lifting δ under $\eta(z) = z^n$ is n-Z-orientable. It follows that ψ^{-1} of such a map still n-Z-orientable.

Let us now observe that the condition for $\mathcal{F}_\delta$ to be n-Z-orientable means that the tessellation $\mathcal{T}_{K,P_\delta}$ is also n-Z-orientable. It follows that the labelling of the last tessellation is provided by θ_0, that is, that θ_0 is (K, P_δ)-admissible. As the induced labelling from $\mathcal{T}_{K,P_\delta}$ on $\mathcal{T}_{K,P_\delta,\Gamma} = \mathcal{F}_\delta$ is the one given originally, Γ is (K, P_δ, Γ) is θ-Z-orientable, so it follows that $\Gamma \leq \ker(\theta_0)$. □

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