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On the Dedekind completion and G-module maps

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ACTA Nº **4765**

En Temuco, el día ***miércoles 26 de enero de 2022****, a las* ***12:30*** *horas Chile, se reúnen en sala virtual, los señores Profesores:*

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y habiendo considerado su Trabajo de Tesis titulado

"On the Dedekind completion and G-module maps"

y su defensa correspondiente, ***RESUELVEN:*** Se aprueba la Tesis escrita con nota 7.0 y la exposición con nota 6.8

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CONTENT

Introduction

The concept of G-module was developed by H. Ochsenius and W.H. Schikhof in the late 1990's. It arose as a consequence of the study of Banach spaces over fields with a Krull valuation when the value group has infinite rank. The aim was to give conditions that enabled to generalize in this context the theory of Hilbert spaces.

Previous work had been done in the theory of quadratic forms. Here the emphasis was placed on the Projection Theorem: Every orthogonally closed subspace X is an orthogonal summand of the space E. That is,

$$X = X^{\perp\perp} \quad \rightarrow \quad E = X \oplus X^{\perp} \tag{.1}$$

Such a space, over a field different from $\mathbb{R}$ or $\mathbb{C}$ was described by H. Keller its numerably orthogonal base could not be normalized ([**6**]).

M. P. Soler proved that this was a central requirement. Her theorem [**18**] state that if a space E has an orthonomal base and satisfies (.1) then the base field must be $\mathbb{R}$ or $\mathbb{C}$ and E is a classical Hilbert space.

In [**10**] the problem was studied in the context of a field K with a Krull valuation in which the value group G has infinite rank. The norms of a K-vector space E would be elements of a G-module X. This structure is a linearly ordered set, different from G, and where an action $G \times X \rightarrow X$ is defined. Adequate selection of X ensures that E can never have an orthonomal base. These spaces were termed Norm-Hilbert spaces.

New concepts included G-cyclic modules, morphisms between G-modules, topological types (see [**11**], [**12**], [**13**] and [**14**]). And it is in this context that the questions that are studied in this thesis appear.

The first one refers to the totally ordered group G. Denote by $G^{\#}$ its completion. In [**11**] the authors introduced a set $(G^{\#})_0$ which was proved to be the largest group contained in $G^{\#}$, in [**13**] they gave examples in which $(G^{\#})_0$ was properly contained in $G^{\#}$, and others in which they were equal. The interest lies in the characterization of these cases.

The second one asked for an extension of the results of E. Olivos and W. H. Schikhof in [**14**] where the set of all G-module maps from $G^{\#}$ to $G^{\#}$ were described. Now if X is any G-module, what can be said of the set of all G-module maps $\varphi : X \rightarrow G^{\#}$?

In order to attain these goals, Chapter 1 summarizes the main definitions, properties and theorems about totally ordered groups, convex subgroups, G-modules and G-module maps. This chapter with preliminaries contains the necessary concepts for the development of the work and results in chapters 2 and 3.

Chapter 2 deals with $(G^{\#})_0$. We will present two interesting examples in order to guide the analysis and conclusions. We will determine the necessary and sufficient conditions that G must satisfy in order to have $G \subsetneq (G^{\#})_0$. The convex subgroups of G played a crucial role in this study.

Further, in Chapter 3, we are interested in extending the results in [**14**]. We show that if X is any G-module $M(X, G^{\#})$, the set of all G-module maps $X \rightarrow G^{\#}$, is a totally ordered group. The results in this chapter are a consequence of the ordering of the two subsets of $M(X, G^{\#})$; $M(X, G^{\#})_{\text{sup}}$ and $M(X, G^{\#})_{\text{inf}}$. In addition, it will be essential to know if $\min\{g \in G : g > 1\} = 1$ and to analyze the orbits of X. Finally, we present the next steps for the results of this thesis that could be addressed in future research.

CHAPTER I

Preliminaries

This chapter contains definitions, notations and main properties of a totally ordered group G, G-modules and G-module maps. These concepts are the basis for the study of this thesis: the group $(G^{\#})_0$ and G-module maps over any G-module X. Most of the proofs of the results in this chapter will be referenced to books and classic articles.

We start in the first sections with the theory about totally ordered groups. We are specially interested in convex subgroups, Dedekind completions and the extension of the multiplication of the group G to its completion $G^{\#}$. Later, it is introduced G-module with some examples in order to present, in the last section, the main results about G-module maps.

I.1. Totally ordered groups and convex subgroups

Definition I.1.1.
Let A be a subset of a totally ordered set X. A is called cofinal in X if for all $x \in X$ there is an element $a \in A$ such that $x \leq a$.

Definition I.1.2.
Let $(G, \cdot, \leq)$ be an abelian multiplicatively written group equipped with a total order $\leq$. We will call G a totally ordered group, if $x, y, z \in G$, $x \leq y$ implies $xz \leq yz$.

In a totally ordered group G we say that the order $\leq$ is compatible with the multiplication defined in G. Also,

(1) G is a torsion free group. Indeed, every element of G has infinite order because if there is $g \in G$ with $g > 1$ and $m \in \mathbb{Z}^+$ such that $g^m = 1$ then $1 < g < g^2 < \cdots < g^m = 1$, a contradiction.
(2) If $G \neq \{1\}$ then G has no smallest or largest element. In fact, suppose that $g < 1$ is the smallest element of G then $g^2 \leq g$ and necessarily $g^2 = g$, it is to say $g = 1$, a contradiction.

Example 1. Some typical examples of totally ordered groups are:

(1) $(\mathbb{R}, +, \leq)$, $(\mathbb{R}^+, \cdot, \leq)$, any multiplicative subgroup of $\mathbb{R}^+$ where $\leq$ is the natural ordering on $\mathbb{R}$.

(2) $(\mathbb{R}^+)^2$ with componentwise multiplication and lexicographical ordering, where for $(a, b), (c, d) \in (\mathbb{R}^+)^2$ we have that $(a, b) \leq (c, d)$ if and only if either $a < b$ or $a = b$ and $c \leq d$.

(3) The direct sum $G = \oplus_{i \in \mathbb{N}} G_i$, where for every $i \in \mathbb{N}$, G_i is a totally ordered group and G is equipped with the lexicographical ordering and componentwise multiplication.

(4) Let $(\mathcal{R}, +, \cdot)$ be the Levi-Civita field (see [**17**] for background). For $f \in \mathcal{R}$ with $f \neq 0$, we put $\lambda(f) = \min(supp(f))$ and we define $\lambda(0) = +\infty$. Let $\mathcal{R}^+$ be the set of all non-zero elements $x \in \mathcal{R}$ that satisfy $x[\lambda(x)] > 0$.

$$\mathcal{R}^+ = \{x \in \mathcal{R} : x[\lambda(x)] > 0\} .$$

Let $x, y \in \mathcal{R}$ be given. We say that $y > x$ if $x \neq y$ and $(y - x) \in \mathcal{R}^+$; and we say $y \geq x$ if $y = x$ or $y > x$. Also, we say $y < x$ if $x > y$ and $y \leq x$ if $x \geq y$. With the relation $\geq$, $(\mathcal{R}, +, \cdot)$ becomes a totally ordered field. Furthermore, the order is compatible with the algebraic structure of $\mathcal{R}$, that is, for any x, y, z, we have: $x > y \Rightarrow x + z > y + z$; and if $z > 0$, then $x > y \Rightarrow x \cdot z > y \cdot z$.

Definition I.1.3.
Let C be a subgroup of G. Let $x, y \in C$ and $z \in G$, if $x \leq z \leq y$ implies $z \in C$, we will call C a convex subgroup of G. We denote by Γ_G the set of all convex subgroups of G.

In addition, we have that

(1) Each proper convex subgroup C of G is bounded from below and from above (if we assume otherwise, it leads us to $C = G$).

(2) Γ_G is totally ordered by inclusion. Indeed, let $C_1, C_2 \in \Gamma_G$ such that $C_1 \nsubseteq C_2$. Then there is an element $x \in C_1$ with $1 < x$ and $x \notin C_2$. Pick $y \in C_2$, with $y > 1$. Then $x \nleq y$, otherwise x belongs to the convex subgroup C_2. Thus $1 < y < x$ and therefore $y \in C_1$ because C_1 is convex. We conclude that $C_2 \subset C_1$.

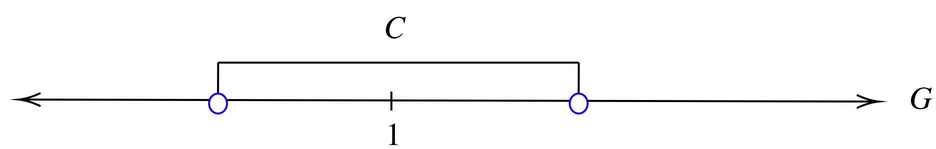


FIGURE I.1. A convex subgroup C of a totally ordered group G can be represented as an open interval of elements in G.

Example 2.

(1) Every totally ordered group $G \neq \{1\}$ contains two trivial convex subgroups, namely $\{1\}$ and G.

(2) $C = \{1\} \times \langle 3 \rangle$ is the unique proper convex subgroup of the totally ordered group $G = \langle 2 \rangle \times \langle 3 \rangle$ with the lexicographical order and componentwise multiplication (see Figure I.2).

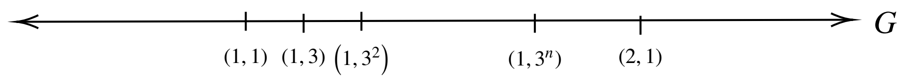


FIGURE I.2. for all $n \in \mathbb{Z}$, $(1, 3^n) < (2, 1)$

(3) Let $G \subset \mathbb{Q} \times \mathbb{Q}$ be the additive group generated by the vectors $\left(p_n^{-1}, np_n^{-1}\right)$, $n = 1, 2, 3, ...$, where p_n is the nth prime number, i. e., $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$, and let G be lexicographically ordered and with componentwise addition.

The set $C = \{0\} \times \mathbb{Z}$ is a convex subgroup of G. Indeed, for all $b \in \mathbb{Z}$, the element $(0, b) \in C$ can always be written as

$$(0, b) = 3b\left(\frac{1}{3}, \frac{2}{3}\right) - 2b\left(\frac{1}{2}, \frac{1}{2}\right) \in G$$

Now, let $n \in \mathbb{N}$ and $(a, b) \in G$ such that $(0, 0) \leq (a, b) \leq (0, n)$. This implies that $a = 0$, because the order of the elements in G is lexicographic. On the other hand, by the definition of C there are integers $n_1, n_2, ..., n_k$ such that

$$(0, b) = n_1\left(\frac{1}{p_{i_1}}, \frac{i_1}{p_{i_1}}\right) + n_2\left(\frac{1}{p_{i_2}}, \frac{i_2}{p_{i_2}}\right) + \cdots + n_k\left(\frac{1}{p_{i_k}}, \frac{i_k}{p_{i_k}}\right),$$

hence $0 = \dfrac{n_1}{p_{i_1}} + \dfrac{n_2}{p_{i_2}} + \cdots + \dfrac{n_k}{p_{i_k}}$.

For $j = 1, \ldots, k$ let

$$P_j = p_{i_1} \cdot p_{i_2} \cdots p_{i_{j-1}} \cdot p_{i_{j+1}} \cdots p_{i_k},$$

and thus $0 = n_1P_1 + n_2P_2 + \cdots + n_kP_k$ or

$$n_jP_j = -\left(n_1P_1 + \cdots n_{j-1}P_{j-1} + n_{j+1}P_{j+1} + \cdots n_kP_k\right).$$

As for all $j = 1, \ldots, k$, p_{i_j} divides P_r if and only if $j \neq r$, we deduce that $n_j = \lambda_j p_{i_j}$, with $\lambda_j \in \mathbb{Z}$ for all j. So, $b = i_1\lambda_1 + \cdots + i_j\lambda_j + \cdots i_k\lambda_k \in \mathbb{Z}$, i. e., $(0, b) \in C$. For details see ([**3, 19**]).

Definition I.1.4.
A convex subgroup C is called principal if there is a $g \in G$ such that C is the smallest convex subgroup of G containing g. The order type of the set of all principal nontrivial subgroup is called the rank of G and denoted by rank(G).

Definition I.1.5.
Let K be a field and let G be an ordered group. A Krull valuation on K is a surjective map $|\cdot| : K \to G \cup \{0\}$ satisfying:

(i) $|0| = 0$ *iff* $x = 0$,
(ii) $|x+y| \leq \max\{|x|, |y|\}$,
(iii) $|xy| = |x||y|$,

where 0 is a symbol such that $0 < g$ and $0g = g0 = 0$ for all $g \in G$. The rank of $|\cdot|$ is the rank of G, and G is called the value group of $(K, |\cdot|)$.

Example 3.

(i) Let p be a prime number. The p-adic valuation $|\ |_p$ on $\mathbb{Q}$ is defined by

$$|0|_p = 0 \quad \text{and} \quad |p^n \frac{r}{q}| = \frac{1}{p^n},$$

where $n, r, q \in \mathbb{Z}$, and r, q are not divisible by p. This valuation has rank 1.

(ii) Let us consider $F_0 = \mathbb{R}$ with its usual ordering, and the set of variables $\{X_1, X_2, X_3, \ldots\}$. For $n \in \mathbb{N}$ define $F_n := F_0(X_1, \ldots, X_n)$ and

$$F_\infty := \bigcup_{n=0}^{\infty} F_n$$

F_n is ordered by powers of X_n. A polynomial $p(X_n) = a_r X_n^r + \ldots + a_1 X_1 + a_0 \in F_{n-1}[X_n]$ is positive in F_n if and only if $a_r > 0 \in F_{n-1}$. For $\beta = \dfrac{p(X_n)}{q(X_n)}$ with $p(X_n)$ and $q(X_n)$ in $F_{n-1}(X_n)$ and $q(X_n) \neq 0$, we say that β is a positive element in F_n if and only if $p(X_n)q(X_n) > 0$. Notice that the ordering of F_n extends the ordering of F_{n-1}. F_∞ is an ordered field (see [**7**] for background).

Now, we will define a valuation v on F_∞. First, we describe the value group of v: for every $i \in \mathbb{N}$, let us consider the multiplicative cyclic group G_i generated by $g_i > 1$ and ordered by

$$g_i^r < g_i^t \quad \text{iff} \quad r < t$$

Let G be the direct sum

$$G := \left\{ \gamma = (g_1^{\alpha_1}, g_2^{\alpha_2}, g_3^{\alpha_3}, \ldots) \in \prod_{i=1}^{\infty} G_i : \quad \alpha_i \in \mathbb{Z}, \text{such that } supp(\gamma) \text{ is finite} \right\}$$

where $supp(\gamma) := \{i \in \mathbb{N} : \alpha_i \neq 0\}$. G is a totally ordered group with the componentwise multiplication and antilexicographical ordering.

After that, we define $v : F_\infty \to \Gamma \cup \{0\}$ such that $v|_{\mathbb{R}}$ is the trivial valuation, $v(X_n) := (1, \ldots, 1, g_n, 1 \ldots)$ and 0 is a minimal element such that $0 \cdot g = g \cdot 0 = 0$.

The valuation v is a Krull valuation on F_∞ with rank ω, the first infinite ordinal.

Definition I.1.6. *An ordered group G is called archimedian if for every $a, b \in G$ with $b > 1$, there exists an integer n such that $a < b^n$.*

Proposition I.1.1. *([**1**], Chapter III, 3.2)*
A totally ordered group $(G, \cdot)$ has rank$(G) = 1$ if and only if G is archimedian.

PROOF.

($\Leftarrow$) Let $C \neq \{1\}$ be a convex subgroup of G, and $c \in C$ with $c > 1$. For every $g \in G$ there exists a $n \in \mathbb{Z}^+$ such that $1 \leq |g| \leq c^n$, with $|g| = \max\{g, g^{-1}\}$, hence $g \in C$. Therefore $C = G$.

($\Rightarrow$) Let $g, h \in G \setminus \{1\}$ with $g > 1$. There exists $n \in \mathbb{Z}^+$ such that $1 \leq |h| \leq g^n$. If we assume otherwise, $C = \langle g \rangle$ would be a proper convex subgroup of G, a contradiction.

□

Definition I.1.7. *Two ordered groups G and G' are called order-isomorphic if there exists an isomorphism $f : G \to G'$ such that $a < b \Rightarrow f(a) < f(b)$ for all $a, b \in G$.*

Theorem I.1.1. *([**15**], 1.1 ; [**1**], Chapter III, 3.4)*
A totally ordered group $(G, \cdot)$ has rank$(G) = 1$ if and only if it is isomorphic to a subgroup of $(R^+, \cdot)$.

Thus, for valuations of rank 1 we can always assume that the ordered group G is a subgroup of $(R^+, \cdot)$ with the natural ordering.

Example 4.

(1) Any subgroup of $\mathbb{R}^+$ has rank 1.
(2) The group C in Example 2 has rank 2.
(3) Let $\mathcal{G}$ be the direct sum

$$\mathcal{G} = \bigoplus_{i \in \mathbb{N}} G_i = G_1 \oplus G_2 \oplus G_3 \oplus \ldots$$

where for each $i \in \mathbb{N}$, G_i is an infinite cyclic group generated by $g_i > 1$. Each element in G has the form

$$(g_1^{\alpha_1}, g_2^{\alpha_2}, g_3^{\alpha_3}, \ldots)$$

where $\alpha_i \in \mathbb{Z}$ and $\{i \in \mathbb{N} : \ \alpha_i \neq 0\}$ is finite. With componentwise multiplication and lexicographical order, $\mathcal{G}$ has infinite rank. The subgroup

$$C_1 = \{1\} \oplus G_2 \oplus G_3 \oplus \cdots$$

is the largest non trivial convex subgroup of $\mathcal{G}$. Also,

$$C_2 \quad = \quad \{1\} \oplus \{1\} \oplus G_3 \oplus G_4 \oplus \cdots$$

$$\begin{aligned}
C_3 &= \{1\} \oplus \{1\} \oplus \{1\} \oplus G_4 \oplus G_5 \oplus \cdots \\
&\vdots \\
C_n &= \{1\} \oplus \cdots \oplus \{1\} \oplus G_{n+1} \oplus G_{n+2} \oplus \cdots \\
&\vdots
\end{aligned}$$

are convex subgroups of $\mathcal{G}$ for all $n \in \mathbb{N}$. They are ordered by inclusion

$$\cdots C_{n+1} \subset C_n \subset \cdots \subset C_1 \subset C_0 \subset \mathcal{G}$$

and $\mathcal{G}$ has a decreasing sequence of convex subgroups.

Note that, if we consider the same group $\mathcal{G}$ with componentwise multiplication and *antilexicographical* order, then $\mathcal{G}$ also has rank ω. In this case, the subgroup

$$D_1 = G_1 \oplus \{1\} \oplus \{1\} \oplus \cdots$$

is the smallest non trivial convex subgroup of $\mathcal{G}$,

$$\begin{aligned}
D_2 &= G_1 \oplus G_2 \oplus \{1\} \oplus \{1\} \oplus \{1\} \oplus \cdots \\
D_3 &= G_1 \oplus G_2 \oplus G_3 \oplus \{1\} \oplus \{1\} \oplus \cdots \\
&\vdots \\
D_n &= G_1 \oplus G_2 \oplus \cdots \oplus G_n \oplus \{1\} \oplus \cdots \\
&\vdots
\end{aligned}$$

are convex subgroups of $\mathcal{G}$ for all $n \in \mathbb{N}$ and we have that

$$\{1\} \subset D_1 \subset D_2 \subset \cdots \subset D_n \subset D_{n+1} \subset \cdots \subset \mathcal{G}.$$

Thus, with the antilexicographical order, $\mathcal{G}$ has an increasing sequence of convex subgroups.

(4) Let $(\mathcal{R}, +, \cdot)$ be the Levi-Civita field (see Example 1.4). The multiplicative group $(\mathcal{R}^+, \cdot)$ has infinite rank. In fact, the set

$$\mathcal{L} = \{x \in \mathcal{R}^+ : \lambda(x) = 0\}$$

is the largest convex subgroup of $(\mathcal{R}^+, \cdot)$. The largest proper convex subgroup contained in $\mathcal{L}$ is

$$\mathcal{L}^0 = \{x \in \mathcal{R}^+ : \lambda(x) = 0, x[0] = 1\}.$$

Let $\lambda_1(x) := \min(supp(x) \setminus \{\lambda(x)\})$. For each $r \in \mathbb{Q}^+$, the sets

$$\mathcal{L}^0_r = \{x \in \mathcal{R}^+ : \lambda(x) = 0, x[0] = 1, \lambda_1(x) \geq r\}$$

is a convex subgroup of $(\mathcal{R}^+, \cdot)$.

Proposition I.1.2. *(*[**15**]*, 1. 2)*
If C is a subgroup of G then $\overline{G} = G/C$ is in a natural way a totally ordered group. C is

a convex subgroup of G if and only if the canonical quotient map $\pi : G \to G/C$ is an increasing homomorphism.

Proof.
The quotient map is given by

$$\begin{array}{rcl} \pi : G & \to & G/C \\ g & \mapsto & \overline{g} = gC \end{array}$$

C is a subgroup of the abelian group G and therefore G/C is a group with the multiplication $\overline{g_1} \cdot \overline{g_2} = \overline{g_1 \cdot g_2}$.

Let P be the positive cone of G, that is , $P = \{g \in G : g \geq 1\}$, we will show that $\overline{P} = \pi(P)$ satisfies that $\overline{P} \cap \overline{P}^{-1} = \{1\}$ and for all $\overline{p_1}, \overline{p_2} \in \overline{P}$ then $\overline{p_1 p_2} \in \overline{P}$ and therefore $\overline{P}$ defines an order on G/C.

- since $1 \in P$ we have $\pi(1) = \overline{1} \in \overline{P}$.
- If $\overline{p_1}, \overline{p_2} \in \overline{P}$ then $p_1, p_2 \in P$ and $\overline{p_1} \cdot \overline{p_2} = \overline{p_1 p_2} \in \overline{P}$, because $p_1 p_2 \in P$.
- If $\overline{p} \in (\overline{P} \cap (\overline{P})^{-1})$ then there are $p_1, p_2 \in P$ such that $\pi(p_1) = \pi(p_2)^{-1} = \overline{p}$. Thus $\pi(p_1 p_2) = \overline{1}$ and therefore $p_1 p_2 \in C$. Since C is a convex subgroup and $1 \leq p_1 \leq p_1 p_2$ we have that $p_1 \in C$ and therefore $\overline{p} = \pi(p_1) = \overline{1}$.

Then $\overline{P}$ defines an order in $\overline{G}$ with positive cone $\pi(P) = \overline{P}$, therefore π is an order homomorphism.

Reciprocally, if $\overline{P}$ is a set of positive elements by an order in $\overline{G}$, then for every $c \in C$ and $g \in G$ such that $1 \leq g \leq c$ we have that $1 \leq \pi(g) \leq \pi(c) = \overline{1}$, $\pi(g) = \overline{1}$ and so $g \in C$. Therefore C is a convex subgroup of G.

The order relation defined on $\overline{G}$ is: $\overline{g_1} \leq \overline{g_2}$ in $\overline{G} \Leftrightarrow$ there is $c \in C$ such that $g_1 \leq g_2 c$. □

I.2. Dedekind completion of a Totally Ordered Set

Let A, B be totally ordered sets with $A \subset B$. We say that $s \in B$ is the supremum of A in B and denote s by $\sup_B A$, if s is the smallest upper bound of A in B. Similarly we define $t := \inf_B A$.

Definition I.2.1.
A totally ordered set S is called Dedekind complete if each non-empty and bounded above subset of S has a supremum.

Likewise, we can say that a totally ordered set S is Dedekind complete when every non-empty and bounded below subset of S has an infimum.

In the next proposition, it will be useful to remember the following property about supremum: Let $G \neq \emptyset$ be a totally ordered group and $A, B \neq \emptyset$ two subsets of G such that $\sup A, \sup B \in G$. Then $\sup A \cdot \sup B = \sup(AB)$ where $AB = \{ab : a \in A, b \in B\}$.

Proposition I.2.1.
Let G be a totally ordered group. If $rank(G) > 1$ then G is not Dedekind complete.

Proof.
If $rank(G) > 1$ then G has a proper non trivial convex subgroup C. By the convexity of G, we know that C is bounded (above and below). If we suppose that G is complete, then there exists $s = \sup C \in G$. It follows that

$$s \cdot s = (\sup C)(\sup C) = \sup(C \cdot C) = \sup C = s \in G$$

and so, $s \neq 1$ is an idempotent element of G, a contradiction since G is a group. □

Definition I.2.2.
Let $X \neq \emptyset$ be a totally ordered set. A non empty subset S of C is called a cut if

1. *S is bounded above*
2. *If $x \in S$, $y < x$ then $y \in S$.*
3. *If $\sup_X S$ exists then $\sup_X S \in S$.*

Now, the cuts in X are used for the construction of the completion of X (for details see [**10**] pp. 5). Let $X^\#$ the collection of all cuts of X, and we consider the order by inclusion in $X^\#$. With this order $X^\#$ is a totally ordered set. Let $A \subset X^\#$ be non-empty and bounded above. There is a cut T such that $S \subset T$, for all $S \in A$. Then $V := \cup_{S \in A} S$ is non-empty and bounded above by T, and by adding $\sup_X(V)$ (if it exists) to V we obtain a *cut* equal to $\sup_{X^\#} A$. We have the natural embedding $\varphi : X \to X^\#$ given by $\varphi(x) = \{s \in X : x \leq s\}$. φ is strictly increasing and therefore an order-preserving embedding. $X^\#$ is called the Dedekind completion of X. $X^\#$ is the smallest totally ordered set, Dedekind complete, containing X.

The previous construction is a generalization of the classic construction by cuts of the real numbers (for details see [**16**], appendix of chapter 1).

Some basic properties about $X^\#$ are listed in the next Proposition.

Proposition I.2.2. *(*[**10**], [**4**]*)*
Let X be a totally ordered set and $X^\#$ its completion by cuts. We have the following statements:

(1) X is complete if and only if $X = X^\#$.
(2) X is cofinal and coinitial in $X^\#$.
(3) For every $s \in X^\#$, $\{x \in X : x \leq s\}$ is a cut in X; every cut in X has this form.
(4) If $s, t \in X^\#$, $s < t$ then there exist $x, y \in X$ such that $s \leq x < t$, $s < y \leq t$.
(5) For each $s \in X^\#$, $s = \sup_{X^\#} \{x \in X : x \leq s\} = \inf_{X^\#} \{x \in X : x \geq s\}$.
(6) Let $A \subset X$. If $s = \sup_X A$ then $s = \sup_{X^\#} A$. If $t = \inf_X A$ then $t = \inf_{X^\#} A$.
(7) For each $s \in X$, $s_0 = \max\{x \in X : x < s\}$ exists if and only if $s_1 = \max\{x \in X^\# : x < s\}$ exists. In this case $s_0 = s_1$.
(8) For each $s \in X$, $s = \sup_{X^\#}\{x \in X : x < s\}$ exists if and only if $s = \sup_{X^\#}\{x \in X^\# : x < s\}$.

(9) *For each* $s \in X$, $\sup_{X^\#}\{x \in X : x < s\} = \sup_{X^\#}\{x \in X^\# : x < s\}$.
(10) *The last three statements, are also true if one replace* sup *by* inf.

Proof. The statements (1) to (6) are proved in [**10**] and (7) to (10) in [**4**]. □

In the Section 1.3.2 of [**10**], the completion for $G = \oplus_{i\in\mathbb{N}} G_i$, the direct sum of the groups $G_i =< g_i >$, $i \in \mathbb{N}$, was determined. Indeed, for this group with the componentwise multiplication and antilexicographic order, each $x \in G^\#$ can be written as gs, where $s = sup_{G^\#}(H)$, with H some convex subgroup of G.

I.2.1. Multiplications on $G^\#$. For a totally ordered group G, the extension of the multiplication from G to $G^\#$ is, in general, not unique. There are two canonical multiplication on $G^\#$ which extend the multiplication on G.

Definition I.2.3. *For $x, y \in G^\#$ set*

$$x \bullet y := \sup_{G^\#} \{g_1 g_2 \in G : g_1 \leq x \wedge g_2 \leq y\}$$

$$x \star y := \inf_{G^\#} \{g_1 g_2 \in G : g_1 \geq x \wedge g_2 \geq y\}$$

They are called the dot multiplication and the star multiplication, respectively.

Some properties of the dot and star multiplications are:

Proposition I.2.3. *(*[**11**]*; 1.4.6,* [**13**] *; 3.1)*
Let $x, y, z \in G^\#$ with $y < z$, and let $g \in G$. We have that

(i) $g \bullet x = g \star x$.
(ii) $x \bullet y \leq x \star y$.
(iii) $x \bullet y \leq x \bullet z$ *and* $x \star y \leq x \star z$.
(iv) $y \star x \leq z \bullet x$.
(v) $(G^\#, \bullet)$ *and* $(G^\#, \star)$ *are commutative semigroups with identity element.*

In view of (i) in the Proposition above, for all $g \in G$ and $x \in X$ we denote $gx := g \bullet x = g \star x$. Also, from (ii) the dot and star multiplication, are called the small and large multiplication.

Proposition I.2.4. *(*[**14**]*;4.9)*
The dot multiplication is left continuous i.e. it is continuous as a map

$$\bullet : (G^\#, \tau^l_{G^\#}) \times (G^\#, \tau^l_{G^\#}) \to (G^\#, \tau^l_{G^\#})$$

where $\tau^l_{G^\#}$ is the left order topology. In the same way, the star multiplication is right continuous.

In [**13**] it was proved that if G has infinite rank, there are uncountably many proper extensions of the multiplication of G to its completion $G^\#$. The authors determine new proper multiplications, that is, multiplications $\diamond : G^\# \times G^\# \to G^\#$ that are associative, conmutative, increasing in both variables and extending the multiplication of G. The dot and star multiplications are proper multiplications. The proper multiplications constructed in [**13**] depend on some convex subgroup of G and the authors prove that if Γ_G, the set of all convex subgroups of G, has cardinality κ, then the set of all proper multiplications has cardinality $\geq 2^\kappa$.

Definition I.2.4. *For $x \in G^\#$,* $\mathrm{Stab}(x) := \{g \in G : gx = x\}$

Note that, for all $x \in G^\#$

- $1 \in \mathrm{Stab}(x)$. Moreover if $c \in \mathrm{Stab}(x)$ then $cx = x$ and hence $x = c^{-1}x$; so $c^{-1} \in \mathrm{Stab}(x)$,
- If $c_1, c_2 \in \mathrm{Stab}(x)$ then $c_1c_2x = c_1(c_2x) = c_1x = x$. Thus $c_1c_2 \in \mathrm{Stab}(x)$,
- If $c \in \mathrm{Stab}(x)$, we can suppose that $c > 1$, and let $g \in G$ such that $1 < g < c$. We have that $x \leq gx \leq cx = x$, thus $g \in \mathrm{Stab}(x)$ and $\mathrm{Stab}(x)$ is a convex subgroup of G.
- Let $g \in G$, then $c \in \mathrm{Stab}(gx) \Leftrightarrow c(gx) = gx \Leftrightarrow cx = x \Leftrightarrow c \in \mathrm{Stab}(x)$. Therefore $\mathrm{Stab}(Gx) = \mathrm{Stab}(x)$.

Lemma I.2.1. *(*[**11**]*;1.4.11, 1.4.13,* [**13**]*; 3.4 , 3.5)*
Let $H \subset G$ be a proper convex subgroup of G, let $s := \sup_{G^\#} H, t := \inf_{G^\#} H$. Then

(i) $\mathrm{Stab}(s) = \mathrm{Stab}(t) = H$
(ii) $s \bullet s = s,\ s \bullet t = t,\ t \star t = t,\ s \star t = s$
(iii) *If $x \in G^\#$ is such that $t \leq x \leq s$ and $H \subset \mathrm{Stab}(x)$ then $x = t$ or $x = s$.*
(iv) *For all $x, y \in G^\#$,* $\mathrm{Stab}(x \star y) = \mathrm{Stab}(x \bullet y) = \mathrm{Stab}(x) \cap \mathrm{Stab}(y)$.

If G is not isomorphic to a subgroup of $\mathbb{R}^+$ then $(G^\#, \star)$ and $(G^\#, \bullet)$ are not groups, because the elements s and t in the previous Lemma are idempotents. However, they contain at least one non-trivial group, namely G. We wonder if G is the largest possible.

Definition I.2.5. *Let* $(G^\#)_0 := \left\{x \in G^\# : \mathrm{Stab}(x) = \{1\}\right\}$

In general, for a totally ordered abelian group G, we have the inclusion $G \subseteq (G^\#)_0 \subseteq G^\#$. $(G^\#)_0$ can contain G strictly; for example, if $(G, \cdot)$ is any dense proper subgroup of $(\mathbb{R}^+, \cdot)$ then $G \subsetneq (G^\#)_0 = \mathbb{R}^+$. On the other hand, if $G = \langle g \rangle$ is a cyclic subgroup of $(\mathbb{R}^+, \cdot)$ then $G = (G^\#)_0 = G^\#$. We can find many groups G such that $G \subsetneq (G^\#)_0 \subsetneq G^\#$, for example, if we consider the group $G = \mathbb{Q}^+ \times \mathbb{Q}^+$ with the multiplication by component and lexicographic order, then $G \subsetneq (G^\#)_0 = \mathbb{Q}^+ \times \mathbb{R}^+ \subsetneq G^\#$.

The next Theorem shows that $(G^\#)_0$ is the largest group contained in $G^\#$.

Theorem I.2.1. [**11**]*; 1.4.18*
For each $x \in (G^\#)_0$ the map

$$\begin{array}{rcl} G^\# & \to & G^\# \\ y & \mapsto & y \bullet x = y \star x \end{array}$$

is a bijection which maps $(G^\#)_0$ onto $(G^\#)_0$ and therefore induces a group structure on $(G^\#)_0$, with $x^{-1} = \sup_{G^\#} \{g \in G : gx \leq 1\} = \inf_{G^\#} \{g \in G : gx \geq 1\}$.

Definition I.2.6. *Given a totally ordered group G, we say that it is quasidiscrete if the set $\{g \in G : g > 1\} = 1$ has a smallest element; otherwise G is quasidense.*

In a quasidiscrete group G with $g_0 = \min\{g \in G : g > 1\}$, for all $g \in G$ its sucessor element is gg_0.

Example 5.

(1) Any cyclic subgroup of $\mathbb{R}^+$ is quasidiscrete.
(2) We consider the direct sum in the Example 4

$$\mathcal{G} = \bigoplus_{i \in \mathbb{N}} G_i = G_1 \oplus G_2 \oplus G_3 \oplus \ldots$$

where for each $i \in \mathbb{N}$, G_i is an infinite cyclic group generated by $g_i > 1$ and componentwise multiplication. With antilexicographical order, $\mathcal{G}$ is quasidiscrete and $g_0 = \min\{g \in G : g > 1\} = (g_1, 1, \ldots)$. On other hand, if we consider lexicographical order, then $\mathcal{G}$ is quasidense (this group will be studied in Chapter 2).

The prefix quasi in the previous definition is due to the fact that there are totally ordered groups where $\min\{g \in G : g > 1\}$ exists yet they contain a subgroup for which this is not true. For example, let G be the totally ordered group $\mathbb{R}^+ \times \langle 2 \rangle$ with componentwise multiplication and lexicographic order. G is quasidiscrete because $\min\{g \in G : g > 1\} = (1, 2)$ and it contains the proper subgroup $H = \mathbb{R}^+ \times \{1\}$ which is quasidense sice $\min\{g \in H : g > 1\} = (1, 1)$.

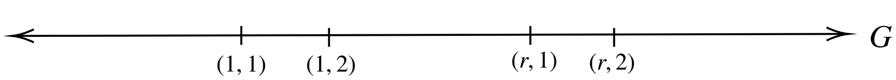


FIGURE I.3. The quasidiscrete group $G = \mathbb{R}^+ \times \langle 2 \rangle$ with componentwise multiplication and lexicographical order. The sucessor of $(1, 1)$ is $(1, 2)$. For all $r \in \mathbb{R}^+$ with $r > 1$, we have that $(1, 2) < (r, 1)$ and the sucessor of $(r, 1)$ is $(r, 2)$.

Proposition I.2.5. *([**11**]; 1.4.11)*
Let $H \subset G$ be a proper convex subgroup, $H \neq \{1\}$, put $t := \inf_{G^\#} H$ and $s := \sup_{G^\#} H$. Then we have

(i) *If G/H is quasidense then $s \star s = s$ and $t \bullet t = t$*
(ii) *If G/H is quasidiscrete then $s \star s = g_0 s > s$, $t \bullet t = g_0^{-1} t < t$ where $g_0 \in G$, $g_0 > s$ and where, with $\pi : G \to G/H$ the canonical map, $\pi(g_0) = \min\{u \in G/H : u > 1\}$*

I.3. Hahn's Theorem

Definition I.3.1. *Let I be a totally ordered set and, for each $i \in I$, let G_i be a totally ordered group written multiplicatively. For all $g = (g_i)_{i\in I} \in \prod_{i\in I} G_i$ where $g_i \in G_i$, for all $i \in I$, the support of g is defined by $supp(g) := \{i \in I : g_i \neq 1\}$.*

Definition I.3.2. *Let I be a totally ordered set and $\{G_i\}_{i\in I}$ be a family of totally ordered groups written multiplicatively, the Hahn product of the family $\{G_i\}_{i\in I}$ is defined by*

$$\mathcal{H}_{i\in I} G_i := \left\{ g \in \prod_{i\in I} G_i : supp(g) \text{ is well-ordered} \right\}$$

$\mathcal{H}_{i\in I} G_i$ is a subgroup of $\prod_{i\in I} G_i$ and, endowed with the lexicographical ordering, it is a totally ordered group (for details see [**5, 15**]).

Theorem I.3.1. *(H. Hahn [**15**]) Every totally ordered group is isomorphic to a subgroup of a Hahn product of copies of $\mathbb{R}$.*

Proposition I.3.1. *([**15**] p.14)*
Let $H \neq \{1\}$ be a convex subgroup of a totally ordered group G. There is a largest convex subgroup H^* such that $H^* \subsetneq H$ and the quotient group H/H^* is a totally ordered group with rank 1.

Let J be a set of indices in one-to-one correspondence with the set of principal convex subgroups of G. We denote by H_j the principal convex subgroup associated to $j \in J$. The index set J is totally ordered with the following rule: for all $j_1, j_2 \in J$, $j_1 \leq j_2 \Leftrightarrow H_{j_2} \subset H_{j_1}$.

For each $j \in J$ let $R_j := H_j/H_j^*$ since, by Proposition I.3.1, R_j has rank 1 it is isomorphic to a subgroup of $\mathbb{R}$.

Definition I.3.3. *For a totally ordered group G the family $(R_j)_{j\in J}$ is called the skeleton of G.*

I.4. G-modules

The structure of G-modules was introduced specifically to serve as a natural range set X for norms defined on a vector space E over a Krull valued field K with value group G.

A generalized norm on E is then defined as a map $\|\cdot\| : E \to X \cup \{0\}$, where 0 is a minimal element adjoined to X, satisfying the following axioms.

(i) $\|x\| = 0 \Leftrightarrow x = 0$
(iI) $\|\lambda x\| = |\lambda|\|x\|$
(iii) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$

for all $x, y \in E$ and $\lambda \in K$.

For instance, the space

$$c_0 := \left\{(x_n)_{n\in\mathbb{N}} \in \prod_{n\in N} K : \lim x_n = 0\right\}$$

is a X-normed space with $\|(x_n)_n\| := \max\{|x_n| : n \in \mathbb{N}\}$. Notice that, $\|\cdot\| : c_0 \to G \cup \{0\}$ where G is the same totally ordered group such that $|\cdot| : K \to G \cup \{0\}$. So, the range set of $\|\cdot\|$ is G.

Throughout this chapter $G = (G, \leq, \cdot)$ is a totally ordered group with unit element 1.

Definition I.4.1.
Let $(X, \leq)$ be a totally ordered set containing at least two elements. It is called a G-module if there exists a map,

$$\begin{aligned} G \times X &\longrightarrow X \\ (g, x) &\longmapsto gx \end{aligned}$$

such that for all $g, g_1, g_2 \in G$, $x, x_1, x_2 \in X$ we have

(i) $g_1(g_2x) = (g_1g_2)x$
(ii) $1x = x$
(iii) $g_1 \geq g_2 \Rightarrow g_1x \geq g_2x$
(iv) $x_1 \geq x_2 \Rightarrow gx_1 \geq gx_2$
(v) *Gx is coinitial in X.*

From the conditions (i)-(v), we can deduce some properties of a G-module X (proofs in [**10**] 1.5.1).

(1) G acts on X and this action preserves the ordering in G and X.
(2) If $x_1 < x_2$ then $gx_1 < gx_2$, for all $g \in G$ and $x_1, x_2 \in X$.
(3) For all $x \in X$, the orbit Gx is cofinal in X.
(4) X has no largest and no smallest elements.

Example 6. ([**10**], 1.5; [**4**])

(a) G is trivially a G-module where the action is simply the multiplication on G.

(b) $G^{\#}$, the completion by cuts of G, is a G-module. with the action $g\alpha := g\bullet\alpha = g\star\alpha$.
(c) Let $\mathcal{D}$ be the divisible closure of G. Let $G^0 := \{d \in \mathcal{D} : d^2 \in G\}$ and consider the map

$$\begin{aligned} \theta : \quad G^0 &\longrightarrow G \\ d &\longmapsto d^2 \end{aligned}$$

The totally ordered set $\sqrt{G} := G^0/Ker(\theta)$ is a G-module with the action $g \cdot \sqrt{k} := \sqrt{g^2 k}$.
(d) If $(G, \cdot, \leq)$ has a non-trivial convex subgroup H, then G/H is a G-module with $g(sH) = (gs)H$.
(e) Let β an ordinal. We consider

$$X := \left\{ x = (x_\alpha)_{\alpha<\beta} \in \prod_{\alpha<\beta} G_\alpha : supp(x) \text{ has an upper bound} \right\}$$

and the antilexicographic order on X. X is a G-module with the action given by $g \cdot x := (gx_\alpha)_{\alpha<\beta}$ for all $g \in G$ and $x \in X$.
(f) Let $(G, \cdot, \leq)$ be a totally ordered group and $G^- := \{g^- : g \in G\}$ a copy of G. We consider $X_2 := G \cup G^-$ and the following rule of order for all $s, t \in G$ such that $t < s$, $t < s^- < s$. We define the action as follows $g \cdot s^- := (gs)^-$. With these definitions, X_2 is a G-module (see Figure I.4).

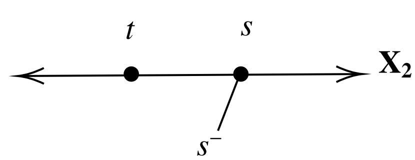


FIGURE I.4. For all $s \in G$, s^- is the predecessor of s. For all $t \in G$ with $t < s$ we have that $t < s^- < s$.

(g) The above example can be generalized as follows. Let β an ordinal and for each ordinal $\alpha < \beta$ let $G^{(\alpha)} := G \times \{\alpha\}$ be a copy of G. Notice that $G^{(\alpha_1)} \cap G^{(\alpha_2)} = \emptyset$ for $\alpha_1 \neq \alpha_2$. Now consider the disjoint union

$$X_\beta := \bigcup_{\alpha<\beta} G^{(\alpha)}$$

with the order defined by

$$(g, \alpha_1) \leq (h, \alpha_2) \Leftrightarrow g < h \text{ or } g = h \text{ and } \alpha_1 \leq \alpha_2.$$

The action of G on X_β is given by $g \cdot (x, \alpha) := (gx, \alpha)$ for all $g \in G$ and $x \in X_\beta$ (see Figure I.5).
(h) A G-module X is called cyclic, if $X =< s >= Gs$ for some element $s \in X$. An arbitrary G-modulo X is the disjoint union of its cyclic submodules

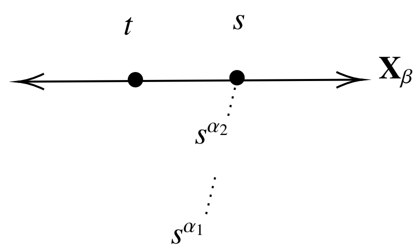


FIGURE I.5. For all $t, s \in G$ with $t < s$ and $\alpha_1 < \alpha_2 \leq \beta$, where $s^{\alpha_1} := (s, \alpha_1)$ and $s^{\alpha_2} := (s, \alpha_2)$

$$X = \mathop{\dot{\bigcup}}_{s_i \in G,\ i \in I} G s_i$$

Conversely, if we have a collection $\{Gs_i, i \in I\}$ of cyclic G-modules, we can extend the ordering on the subsets Gs_i to the union

$$X := \bigcup_{i \in I} G s_i$$

such that X becomes a G-module. For example, we can set a total ordering on I and by declaring that $gs_i > g's_j$ if either $g > g'$ or $g = g'$ and $i > j$.

In the previous chapter (see Definition I.2.4) we introduced the stabilizer of an element x in the G-module $G^{\#}$. The definition easily carries over to arbitrary G-modules.

Definition I.4.2.
Let x be any element in the G-module X. Then we define $\mathrm{Stab}(x) = \{g \in G : gx = x\}$.

Note that $\mathrm{Stab}(x)$ is a proper convex subgroup of G. Indeed, $1 \in \mathrm{Stab}(x)$, and if $g \in \mathrm{Stab}(x)$ then $gx = x$; thus, $x = g^{-1}x$ and $g^{-1} \in \mathrm{Stab}(x)$. Besides, let $u \in G$ such that $g_1 < u < g_2$, with $g_1, g_2 \in \mathrm{Stab}(x)$. Because of the requirement (iv) in the definition of a G-module, we have that $x = g_1 x \leq ux \leq g_2 x = x$, and hence $u \in \mathrm{Stab}(x)$.

On the other hand, if $x_0, x_1 \in X$ and $x_1 \in Gx_0$ then $\mathrm{Stab}(x_0) = \mathrm{Stab}(x_1)$.

Example 7.
Let $x_0 \in X$ and consider the canonical homomorphism $\pi : G \to G/\mathrm{Stab}(x_0)$. The orbit Gx_0 is a $G/\mathrm{Stab}(x_0)$-module with the action $\pi(g)x_0 := gx_0$. Gx_0 has only elements with trivial stabilizer.

Proposition I.4.1. *(*[**10**]*, 1.5.3)*
Let X be a G-module.

(i) *Let $V \subset X$, $g \in G$. If $\sup(V)$ exists then $g\sup(V) = \sup(gV)$. If $\inf(V)$ exists then $g\inf(V) = \inf(gV)$. If V is not bounded above (below) neither is gV.*

(ii) *Let $W \subset G$, $x_0 \in X$. If $\sup_G(W)$ and $\sup_X(Wx_0)$ exist then $sup_G(Wx_0) \leq x_0\sup_G(W)$. If $\inf_G(W)$ and $\inf_X(Wx_0)$ exist then $inf_G(Wx_0) \geq x_0\inf_G(W)$. If W is not bounded above (below) neither is x_0W, and conversely.*

I.5. G-module maps

Definition I.5.1.
Let X, Y be G-modules. A map $\varphi : X \to Y$ is called a G-module map if φ is increasing and $\forall g \in G, x \in X$, $\varphi(gx) = g\varphi(x)$

$M(X, Y)$ is the set of all G-module maps from X to Y, and we put $M(X) := M(X, X)$. The set $M(X, Y)$ can be empty, for example, if X has an element x_0 such that $\mathrm{Stab}(x_0) \neq \{1\}$ then $M(X, G) = \emptyset$. Indeed, we have that for any G-module map φ, $\mathrm{Stab}(x_0) \subset \mathrm{Stab}(\varphi(x_0))$ and hence if $1 \neq g \in \mathrm{Stab}(x_0)$ then $g \in \mathrm{Stab}(\varphi(x_0))$, but $\varphi(x_0) \in G$, a contradiction.

When we consider $Y = G^\#$ we can always determine a G-module map $\phi : X \to G^\#$, where X is any G-module. The following theorem shows this facts, the set $M(X, G^\#)$ is always non empty. We include the demonstration because it is fundamental for the theory of G-module maps.

Theorem I.5.1. *(*[**10**]*; 1.5.6)*
Let X be a G-module. Then there exists a G-module map $\phi : X \to G^\#$.

Proof.
Let x_0 be any element in the G-module X. We know that Gx_0 is coinitial in X and $G^\#$ is complete, therefore we can set

$$\begin{aligned} \phi : \quad & X \longrightarrow G^\# \\ & x \longmapsto \sup_{G^\#}\{g \in G : gx_0 \leq x\} \end{aligned}$$

Obviously, if $x_1 \leq x_2$ then $\phi(x_1) \leq \phi(x_2)$, for all $x_1, x_2 \in X$. Also, we have that for all $h \in G$,

$$\begin{aligned} \phi(hx) &= \sup_{G^\#}\{g \in G : gx_0 \leq hx\} \\ &= \sup_{G^\#}\{g \in G : h^{-1}gx_0 \leq x\} \\ &= \sup_{G^\#}\{hu \in G : ux_0 \leq x\} \quad \text{with } u = h^{-1}g \\ &= h\sup_{G^\#}\{u \in G : ux_0 \leq x\} \\ &= h\phi(x) \end{aligned}$$

As Gx_0 is cofinal in X, we can also define $\phi(x) = \inf_{G^\#}\{g \in G : gx_0 \geq x\}$. □

In the study of X-normed Banach space $(E, \| \|)$, in [**12**] the authors use this theorem and extend a G-module map $\phi : X \to G^{\#}$ to a map $\phi : X \cup \{0\} \to G^{\#} \cup \{0\}$ to define a norm $\| \|_{\phi}$ on $G^{\#}$. With this new norm, $(E, \| \|_{\phi})$ is a Banach space. They use these two norms for comparing two Norm Hilbert Spaces and their operators.

In [**14**], all G-module maps in $M(G^{\#})$ were determined. For this, in $G^{\#}$ they consider two topologies stronger than the order topology: the left order topology, τ^l, generated by the intervals $(x, y] := \left\{z \in G^{\#} : x < z \leq y\right\}$ where $x, y \in G^{\#}$ and $x < y$ and the right order topology, τ^r, generated by the intervals $[x, y) := \left\{z \in G^{\#} : x \leq z < y\right\}$ where $x, y \in G^{\#}$ and $x < y$.

Definition I.5.2.
Let $f : A \to B$ a map with A, B totally ordered sets. We say that f is left continuous, if $f : (A, \tau^l) \to (B, \tau^l)$ is continuous. In the same way, we define right continuity.

In [**14**] we find a description of the following sets of G-module maps:

- $M(G^{\#}) = \{\varphi : G^{\#} \to G^{\#} : \varphi$ is a G-module map$\}$
- $M^l(G^{\#}) = \{\varphi \in M(G^{\#}) : \varphi$ is left continuous$\}$
- $M^r(G^{\#}) = \{\varphi \in M(G^{\#}) : \varphi$ is right continuous$\}$

The next Theorem uses the fact that the dot multiplication $\bullet$ and the star multiplication $\star$ are left and right continuous respectively (see Chapter I Definition I.2.3, Proposition I.2.4, [**14**] and [**13**]).

Theorem I.5.2. *([**14**]; 5.1,5.2)*
Let $\varphi \in M(G^{\#})$. Then

(i) $\varphi \in M^l(G^{\#})$ *if and only if it has the form* $x \mapsto x \bullet \alpha$ *for some* $\alpha \in G^{\#}$.
(ii) $\varphi \in M^r(G^{\#})$ *if and only if it has the form* $x \mapsto x * \alpha$ *for some* $\alpha \in G^{\#}$.
(iii) $M(G^{\#}) = M^l(G^{\#}) \bigcup M^r(G^{\#})$.

The previous theorem shows that each G-module map in $M(G^{\#})$ is left continuous or right continuous.

Definition I.5.3.
We define an order relation $\leq$ on $M(X, Y)$ by $\varphi_1 \leq \varphi_2$ if only if $\varphi_1(x) \leq \varphi_2(x)$ for all $x \in X$.

The next theorem shows that this order is total on $M(G^{\#})$.

Theorem I.5.3. *([**14**]; 5.3)*
$M(G^{\#})$ is a totally ordered set.

So, for all $\varphi_1, \varphi_2 \in M(G^\#)$ we have that $\varphi_1 \leq \varphi_2$ or $\varphi_2 \leq \varphi_1$. We show the different possible cases in the next Corollary.

Corollary I.5.1.
Let $\varphi_1, \varphi_2 \in M(G^\#)$ and $\alpha, \beta \in G^\#$ with $\alpha < \beta$. We have the following cases:

(i) *If $\varphi_1(x) = x \bullet \alpha$, $\varphi_2(x) = x \bullet \beta$; then $\varphi_1 \leq \varphi_2$.*
(ii) *If $\varphi_1(x) = x \star \alpha$, $\varphi_2(x) = x \star \beta$; then $\varphi_1 \leq \varphi_2$.*
(iii) *If $\varphi_1(x) = x \star \alpha$, $\varphi_2(x) = x \bullet \beta$; then $\varphi_1 \leq \varphi_2$.*
(iv) *If $\varphi_1(x) = x \bullet \alpha$, $\varphi_2(x) = x \star \beta$; then $\varphi_1 \leq \varphi_2$.*

Proof.

(i) By the result of Proposition I.2.3 in Chapter I, we have that for all $x \in G^\#$ $x \bullet \alpha \leq x \bullet \beta$ and therefore $\varphi_1(x) \leq \varphi_2(x)$, so $\varphi_1 \leq \varphi_2$.
(ii) Similarly, we have that $x \star \alpha \leq x \star \beta$ for all $x \in X$, then $\varphi_1 \leq \varphi_2$.
(iii) By Proposition I.2.3 (iv) in Chapter I, we have that if $\alpha \leq \beta$ then $x \star \alpha \leq x \bullet \beta$ for all $x \in G^\#$. Thus, $\varphi_1(x) = x \star \alpha \leq x \bullet \beta = \varphi_2(x)$ for all $x \in G^\#$, that is to say $\varphi_1 \leq \varphi_2$.
(iv) We have that $\varphi_1(x) = x \bullet \alpha \leq x \bullet \beta$ and hence by I.2.3 (ii), $x \bullet \beta \leq x \star \beta$ for all $x \in X$, so $x \bullet \alpha \leq x \star \beta$. Therefore $\varphi_1 \leq \varphi_2$.

□

Theorem I.5.4. *([**14**]; 5.4)*
$M(G^\#)$ is a G-module with the action $g \cdot \varphi := (g\varphi)$ where for all $x \in X$, $(g\varphi)(x) := \varphi(gx)$.

Proof.
The first two requirements in the definition of a G-module are clear from the definition of the action of G on $M(G^\#)$. We will show (*iii*), (*iv*) and (*v*) (see Definition I.4.1 in the section I.4). Let $g_1, g_2 \in G$ and $\varphi, \varphi_1, \varphi_2 \in M(G^\#)$. Then

(iii) If $g_1 < g_2$ then $g_1 x \leq g_2 x$ for all $x \in G^\#$, because $G^\#$ is a G-module. Also, since φ is a G-module map, we have that $\varphi(g_1 x) \leq \varphi(g_2 x)$, thus $g_1\varphi(x) \leq g_2\varphi(x)$ for all $x \in G^\#$, and hence $g_1\varphi \leq g_2\varphi$.
(iv) $\varphi_1(x), \varphi_2(x) \in G^\#$ for all $x \in G^\#$. If $\varphi_1(x) < \varphi_2(x)$, then since $G^\#$ is a G-module, for all $x \in G^\#$ we have that $g\varphi_1(x) \leq g\varphi_2(x)$ and therefore $g\varphi_1 \leq g\varphi_2$.
(v) Let φ_1 be a G-module map with $\varphi_1(x) = x \bullet \beta$ and $\beta \in G^\#$. We will prove that the orbit $G\varphi_1$ is cofinal in $M(G^\#)$, that is, for all $\varphi(x) = x \bullet \alpha$ with $\alpha \in G^\#$ we can find $g \in G$ such that $\varphi \leq g\varphi_1$. Indeed, $G\beta$ is cofinal in $G^\#$, because $G^\#$ is a G-module, then we can always find a $g_\alpha \in G$ such that $\alpha \leq g_\alpha\beta$ and by Theorem I.5.3 and Corollary I.5.1 we have that for all $x \in G^\#$,

$$\varphi(x) = x \bullet \alpha \leq x \bullet (g_\alpha\beta) = g_\alpha(x \bullet \beta) = g_\alpha\varphi_1(x).$$

Therefore $\varphi \leq g_\alpha\varphi_1$.

Similarly, if $\varphi(x) = x \star \alpha$ then we can find $g_\alpha \in G$ such that $\alpha \leq g_\alpha \beta$ and by Corollary I.5.1 , $x \in G^\#$,

$$\varphi(x) = x \star \alpha \leq x \bullet (g_\alpha \beta) = g_\alpha (x \bullet \beta) = g_\alpha \varphi_1(x).$$

The same argument applies if $\varphi_1(x) = x \star \beta$.

$\square$

CHAPTER II

The Largest Group Contained in $G^\#$

Let G be a totally ordered group written multiplicatively and $G^\#$ be its Dedekind completion, that is, the completion by cuts. In the Preliminaries we saw that it is posible to extend the multiplication on G to a multiplication for elements in $G^\#$ in many ways [**13**]. Two canonical extensions were presented: the *dot* $\bullet$ *and star* $\star$ *multiplications*. In addition, if G is not isomorphic to a subgroup of $\mathbb{R}^+$ then $(G^\#, \bullet)$ and $(G^\#, \star)$ are not groups, but they contain at least one non-trivial subgroup, G.

In Proposition 1.4.18 in [**11**], the authors show that

$$(G^\#)_0 = \left\{x \in G^\# : Stab(x) = \{1\}\right\}$$

is the largest group contained in $G^\#$, in which for all $x \in (G^\#)_0$ and $y \in G^\#$, $x \bullet y = x \star y$.

Basic examples show that G and $(G^\#)_0$ may coincide or differ. For instance, if G is any cyclic subgroup of $(\mathbb{R}^+, \cdot)$ then $G = (G^\#)_0 = G^\#$, yet if $G = \mathbb{Q}^+$ then $G \subsetneq (G^\#)_0 = G^\# = \mathbb{R}^+$. In this case, both groups are of rank 1. An example of a totally ordered group with rank greater than 1 is the group in Example 2.2 of [**13**], the direct sum $G = \bigoplus_{i\in\mathbb{N}} G_i$, where each G_i is a multiplicative copy of $\mathbb{Z}$ with componentwise multiplication and antilexicographical ordering. In this example $(G^\#)_0 = G$, because each $x \in G^\#$ can be written as gs , where $g \in G$, and s is the supremum of some convex subgroup (see [**10**] Example 1.3.2.). Thus, whether or not $G = (G^\#)_0$ does not depend on the rank. Our aim in this chapter is to determine the conditions for $G \subsetneq (G^\#)_0$.

In the Section II.1 we give a necessary condition for the strict inclusion $G \subsetneq (G^\#)_0$. Following this we shall study two non-trivial examples of totally ordered groups with a decreasing sequence of convex subgroups. We determine its convex subgroups and some useful properties of them. In the last section we establish a sufficient condition on G that ensures that $G \subsetneq (G^\#)_0$.

II.1. A necessary condition for $G \subsetneq (G^\#)_0$

The following definition gives us a classification of totally ordered groups and allows us to determine a sufficient condition for $G = (G^\#)_0$.

Definition II.1.1.
Given a totally ordered group G, we say that it is quasidiscrete if the set $\{g \in G : g > 1\}$ has a minimal element; otherwise G is quasidense.

The prefix quasi in the previous definition is due to the fact that there are totally ordered groups where $\min\{g \in G : g > 1\}$ exists yet they contain a subgroup for which this is not true. Note that, if G is a cyclic subgroup of $\mathbb{R}^+$, then G is quasidiscrete and $G = (G^{\#})_0$. The same happens with the direct sum $G = \bigoplus_{i \in \mathbb{N}} G_i$, where each G_i is a multiplicative copy of $\mathbb{Z}$ with componentwise multiplication and antilexicographical ordering (Example 2.2 in [**13**]), where the successor of 1_G is the element $(g_1, 1, \ldots)$, G is quasidiscrete and $G = (G^{\#})_0$. This behavior of $(G^{\#})_0$ was reported in [**9**], but the proof was not included. We present one in the following Lemma.

Lemma II.1.1. *If G is quasidiscrete then $(G^{\#})_0 = G$.*

Proof. It is enough to prove that $(G^{\#})_0 \subset G$. Suppose that there is $s \in (G^{\#})_0$ with $1 < s$ and $s \notin G$. Thus, by definition, $Stab(s) = \{1\}$.

Now, let $g_0 = \min\{g \in G : g > 1\}$. We have that $1 < g_0 < s$, since if $s < g_0$ then, by Proposition I.2.2 (*iv*), there would exist $g \in G$ such that $1 < g < s < g_0$, which would contradict our choice of g_0 .

Now, for all $g \in G$ with $g > 1$ we have that $1 < g_0 \leq g$; then, multiplying by s, we obtain that

$$1 < g_0 < s \leq g_0 s \leq sg.$$

But $s \neq g_0 s$ since $Stab(s) = \{1\}$; it follows that $1 < g_0 < s < g_0 s \leq sg$.

Now, let $u \in G$ with $u < s$. Then $1 < u^{-1}s$ and, by the definition of g_0, we have that $1 < g_0 < u^{-1}s$, so $u < g_0^{-1}s$ for all $u < s$. Therefore, as $s = \sup_{u \in G}\{u < s\}$ then $s \leq g_0^{-1}s$ or $g_0 s \leq s$, which contradicts the fact that $s < g_0 s$ shown above. □

Lemma II.1.1 gives a necessary condition for $G \subsetneq \left(G^{\#}\right)_0$, that is, G must be quasidense. However this condition is not sufficient. We will show this in the next example.

Example 8.
We consider $G = \langle 2 \rangle \times \mathbb{R}^+$ with the lexicographical order and componentwise multiplication. Firstly, note that the sequence of elements of G, $\left(1, \frac{n+1}{n}\right)_{n \in \mathbb{N}}$ is decreasing and converges to 1_G and therefore $\min\{g \in G : g > 1\} = 1_G$, i. e. G is a quasidense group.

Secondly, we know that $H = \{1\} \times \mathbb{R}^+$ is a convex subgroup of G and $s = \sup\{H\}$ or $t = \inf\{H\}$ are not in $(G^{\#})_0$ because $Stab(s) = Stab(t) = H$ (see Preliminaries Lemma I.2.1).

Next, if $\alpha \in G^{\#} \setminus G$ then $\alpha = \sup(A)$, with A some bounded above subset of G.

$$\begin{aligned} \alpha &= \sup_{G^{\#}}\{g \in G : g < \alpha\} \\ &= \sup_{G^{\#}}\{(2^n, r) \in A : n \in \mathbb{Z}, r \in \mathbb{R}^+, (2^n, r) < \alpha\}. \end{aligned}$$

Because A is bounded , there exists $m \in \mathbb{Z}$ such that $(2^m, r) \in A$ and $(2^{m+1}, r) > \alpha$. Note that, $(2^m, r) \in A$ for all $r \in \mathbb{R}$. Indeed, if we suppose that there exists $q \in \mathbb{R}^+$ such that $\alpha < (2^m, q)$ then for $(2^m, r) \in A$,

$$(2^m, r) < \alpha < (2^m, q),$$

multiplying this inequality by $(2^{-m}, 1) \in G$,

$$(1, r) < \alpha < (1, q).$$

This implies that $\alpha \in H$, the convex subgroup of G, a contradiction.

Finally,

$$\begin{aligned}
\alpha &= \sup_{G^\#}\{g \in G : g < \alpha\} \\
&= \sup_{G^\#}\{(2^m, r) : r \in \mathbb{R}^+\} \\
&= \sup_{G^\#}\{(2^m, 1) \cdot (1, r) : r \in \mathbb{R}^+\} \\
&= (2^m, 1) \cdot \sup_{G^\#}\{(1, r) : r \in \mathbb{R}^+\} \\
&= (2^m, 1) \cdot s.
\end{aligned}$$

Therefore, $(G^\#)_0 = G = \langle 2 \rangle \times \mathbb{R}^+$ and G is a quasidense group.

In short, for a totally ordered group G written multiplicatively, we have the inclusion $G \subseteq (G^\#)_0 \subseteq G^\#$. The strict inclusion $G \subsetneq (G^\#)_0$ does not depend neither on $rank(G)$ nor on G being quasidense.

Corollary II.1.1. *If G contains a first non trivial cyclic convex subgroup , then $G = (G^\#)_0$.*

Proof. Let $H_1 = \langle h \rangle$, generated by $h > 1$, be the first convex subgroup of G. This implies that $h = \inf\{g \in G : g > 1\}$, so G is quasidiscrete and therefore $G = (G^\#)_0$. □

For the aforementioned reason, we are interested in groups with a decreasing sequence of convex subgroups. We shall study in detail two ordered groups with these characteristics in the following two examples.

II.2. Example: Lexicographic Direct Sum

Let $(G_i)_{i \in \mathbb{N}}$ be a family of infinite cyclic groups, where $G_i = (\langle g_i \rangle, \cdot)$ and $g_i > 1$. Each group $G_i = \left\{g_i^\alpha : \alpha \in \mathbb{Z}\right\}$ becomes a totally ordered group by the order relation $g_i^\alpha < g_i^\beta \Leftrightarrow \alpha < \beta$.

Now, let $\mathcal{G}$ be the direct sum of the groups G_i

$$\mathcal{G} = \bigoplus_{i\in\mathbb{N}} G_i = G_1 \oplus G_2 \oplus G_3 \oplus \cdots$$

An element $g \in \mathcal{G}$ will be written as $g = \left(g_i^{\alpha_i}\right)_{i\in\mathbb{N}} = \left(g_1^{\alpha_1}, g_2^{\alpha_2}, g_3^{\alpha_3}, \ldots\right)$ where $\alpha_i \in \mathbb{Z}$ for all $i \in \mathbb{N}$ and $|supp(g)| = |\{i \in \mathbb{N} : \alpha_i \neq 0\}| < \infty$. With the componentwise multiplication and the lexicographic order, $\mathcal{G}$ becomes a totally ordered group. Hence, if $f = (g_i^{\alpha_i})_{i\in\mathbb{N}}$, $h = (g_i^{\beta_i})_{i\in\mathbb{N}}$ with $h \neq f$ and $r = \min\{i \in \mathbb{N} : \alpha_i \neq \beta_i\}$ then $f < h \Leftrightarrow \alpha_r < \beta_r$.

If we denote by e_k the element $(g_i^{\gamma_i})_{i\in\mathbb{N}} \in \mathcal{G}$ such that $\gamma_i = 0$ for all $i \neq k$ and $\gamma_k = 1$, then

$$e_1 = (g_1, 1, 1, \cdots) \quad e_2 = (1, g_2, 1, \cdots) \quad e_3 = (1, 1, g_3, 1, \cdots) \quad \cdots$$

and each element $g \in \mathcal{G}$ can be written as the product

$$g = e_1^{\alpha_1} e_2^{\alpha_2} e_3^{\alpha_3} \cdots = \prod_{i\in\mathbb{N}} e_i^{\alpha_i},$$

where the set $\{i : \alpha_i \neq 0\}$ is bounded.

Using this notation, we will make a tree diagram of $\mathcal{G}$, as shown in Figure II.1.

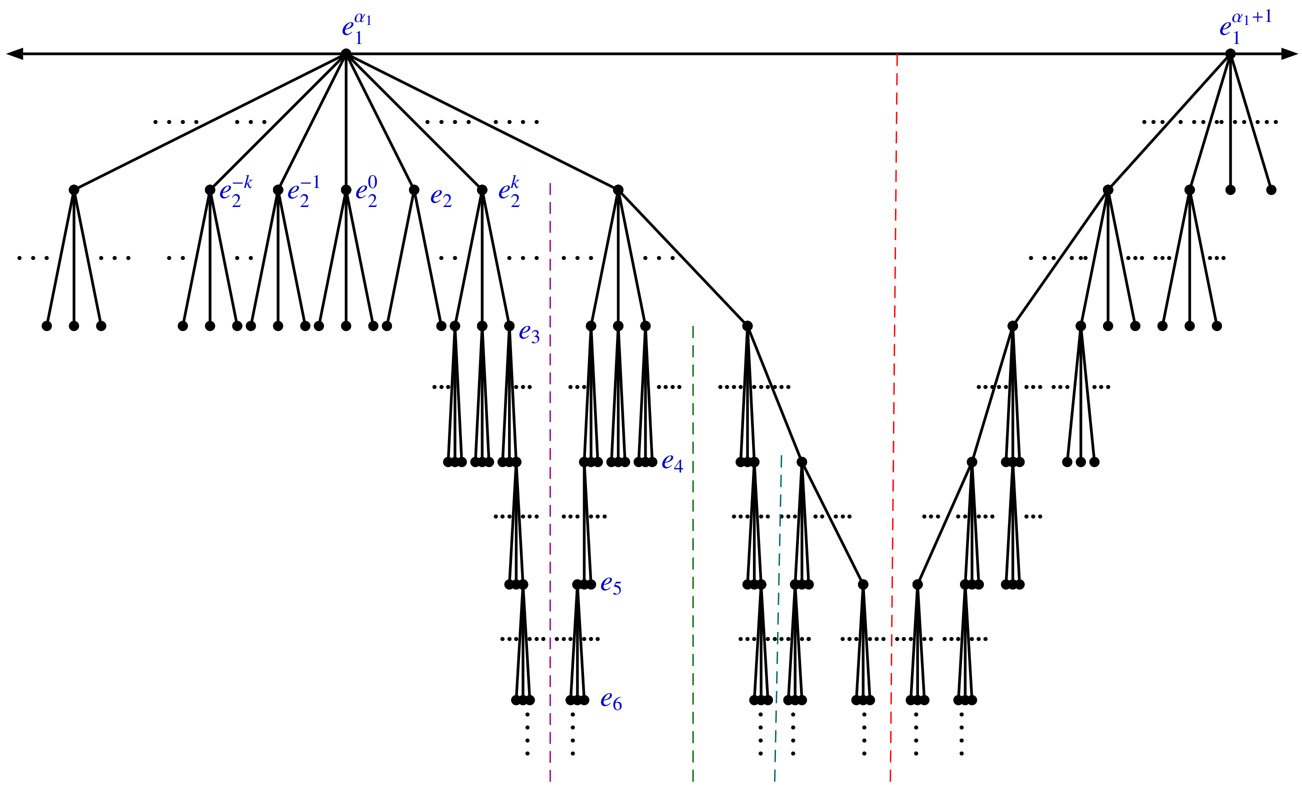


FIGURE II.1. Tree representation for the totally ordered group $\mathcal{G}$

The levels in the tree diagram, from top to bottom are numbered from one downwards and represent the different options for the components $e_i^{\alpha_i}$ of an element $g = (e_i^{\alpha_i})_{i\in\mathbb{N}} \in \mathcal{G}$. Each possible path represents one element in $\mathcal{G}$ and its location in the tree indicates its position in $\mathcal{G}$.

This scheme allows us to visualize some notable subsets and properties of $\mathcal{G}$. For example, given two paths in the tree, the element in $\mathcal{G}$ represented by the path located in a branch further to the right is larger than the element represented by the path located on the left.

II.2.1. Two outstanding subsets of $\mathcal{G}$. Consider the following subsets of $\mathcal{G}$. For $i \in \mathbb{N}$ and $g \in \mathcal{G}$, we define

$$A_g^i := \{ge_i^\nu : \nu \in \mathbb{Z}\}.$$

as well as

$$C_g^i := \left\{h = ge_i^{\nu_i} e_{i+1}^{\nu_{i+1}} \cdots e_r^{\nu_r} : r \in \mathbb{N},\ i \le r,\ \nu_j \in \mathbb{Z},\ i \le j \le r\right\}$$

We can make a tree diagram of A_g^i and C_g^i as shown by the Figures II.2 and II.3.

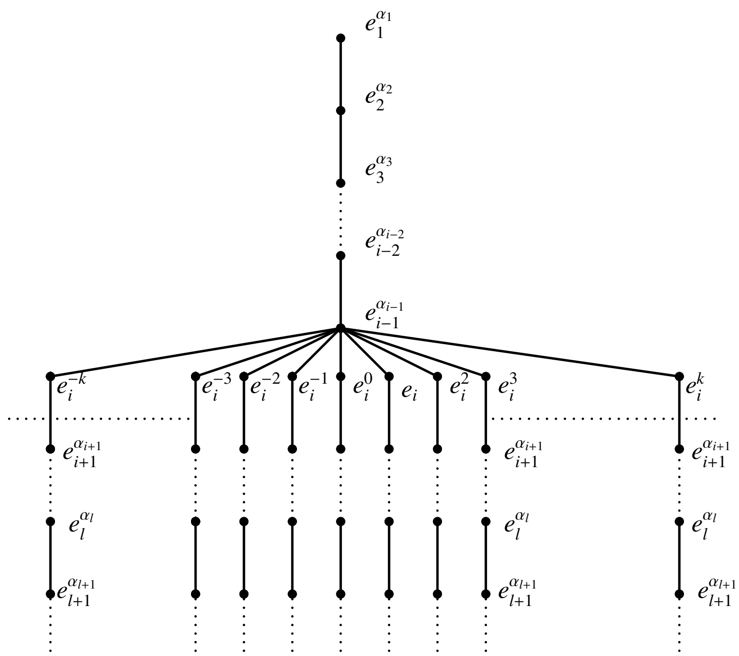


FIGURE II.2. The branch in the figure represents the set A_g^i

Example 9. As an example to describe the tree diagram and subsets A_g^i and C_g^i, we consider the totally ordered group $G = \langle 2\rangle \times \langle 3\rangle \times \langle 5\rangle \times \langle 7\rangle$ with componentwise multiplication and lexicographic order.

Let $i = 3$ and let $g = (2, 3, 1, 7^2) \in G$. Then, we have that

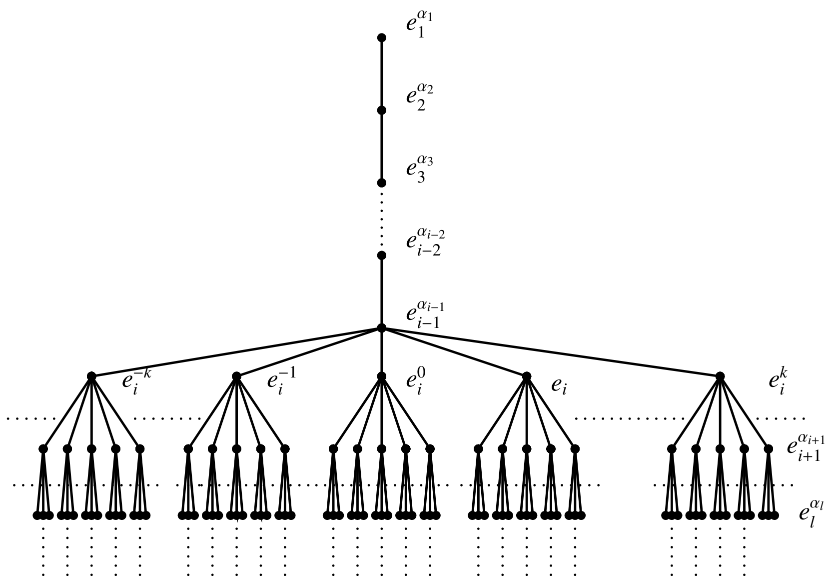


FIGURE II.3. The branch in the figure represents the set C_g^i

$$e_1 = (2,1,1,1), \quad e_2 = (1,3,1,1), \quad e_3 = (1,1,5,1), \quad e_4 = (1,1,1,7),$$

so $g = (2,3,1,7) = (2,1,1,1) \cdot (1,3,1,1) \cdot (1,1,5,1)^0 \cdot (1,1,1,7)^2$ and

$$\begin{aligned} A_g^3 &= \{ g e_3^{\nu} : \nu \in \mathbb{Z} \} \\ &= \left\{ (2,3,5^{\nu},7^2) : \nu \in \mathbb{Z} \right\} \end{aligned}$$

and

$$\begin{aligned} C_g^3 &= \left\{ g e_3^{\nu_3} e_4^{\nu_4} : \nu_j \in \mathbb{Z},\ i \le j \right\} \\ &= \{ (2,3,5^{\nu_3},7^{\nu_4}) : \nu_3, \nu_4 \in \mathbb{Z} \}. \end{aligned}$$

The following figures show the element $g = (2,3,1,7^2)$ and the subsets A_g^3 and C_g^3 in the tree diagram for the group G.

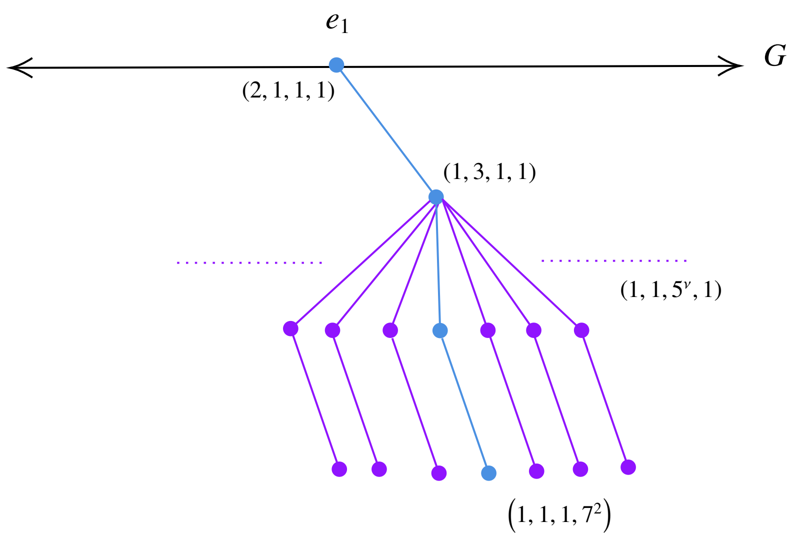


FIGURE II.4. The subset A_g^3 where $g = (2, 3, 1, 7^2)$

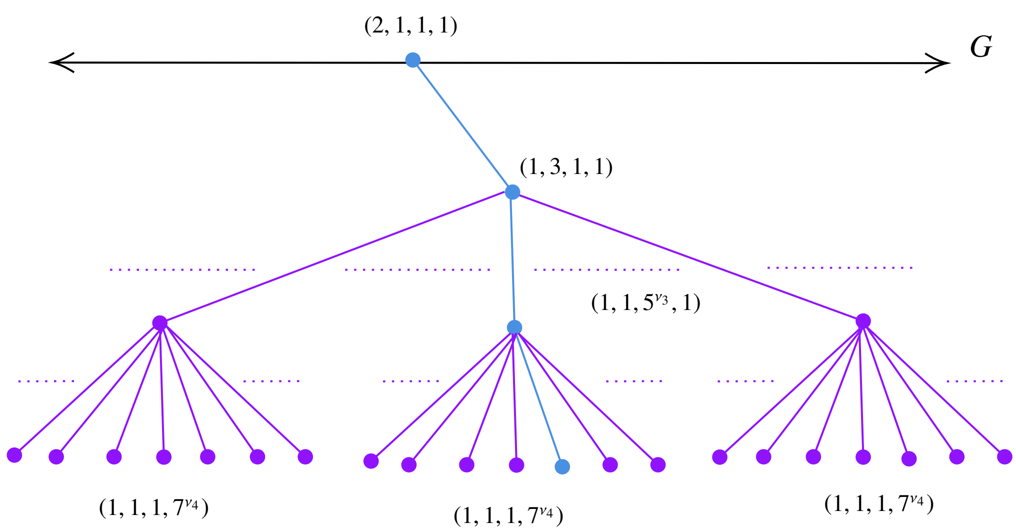


FIGURE II.5. The subset C_g^3 where $g = (2, 3, 1, 7^2)$

The next Proposition describes important properties of the subsets A_g^i and C_g^i.

Proposition II.2.1.
Let $g, h \in \mathcal{G}$ and $1 < i \in \mathbb{N}$.

(i) *The group $\mathcal{G}$ is quasidense.*
(ii) *The sets A_g^i and C_g^i are bounded.*

(iii) $\sup C_g^i$ *and* $\inf C_g^i$ *do not belong to* $\mathcal{G}$.
(iv) $\sup C_g^i = \sup A_g^i$.
(v) *Given* $k > 1$ *with* $k \neq i$, *we have* $\sup A_g^i \neq \sup A_g^k$.

Proof.

(i) If we suppose that the set $\{g \in \mathcal{G} : g > 1\}$ has a first element,

$$g_0 = (g_i^{\alpha_i})_{i \in \mathbb{N}} = \min \{g \in \mathcal{G} : g > 1\}$$

then there exists an element $u = g_0 e_{k+1}^{-1} \in \mathcal{G}$ where $k = \min\{i \in \mathbb{N} : \alpha_i \neq 0\}$ which satisfies $1 < u < g_0$, a contradiction (see Figure II.6).

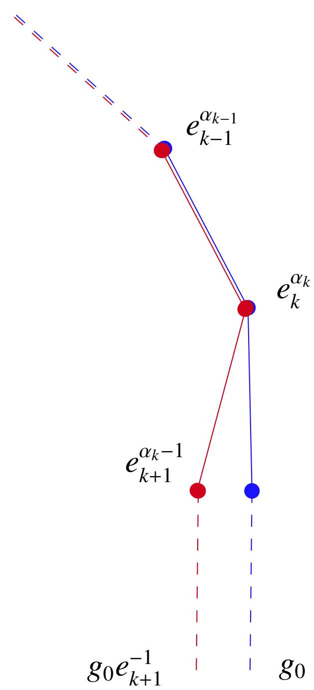


Figure II.6. The blue path represents g_0 and the red path the element u.

(ii) Basically, it can be seen in the diagrams the procedure that gives bounds for any of these sets. We locate the set in the diagram and we move to the right for upper bounds and to the left for lower bounds. As an example, the element ge_{i-1} is an upper bound and ge_{i-1}^{-1} is a lower bound of A_g^i and C_g^i (see Figure II.7).

(iii) The $\sup C_g^i$ must satisfy that for all $n_j \in \mathbb{Z}$, $j \geq i$

$$(g_1^{\alpha_1}, g_2^{\alpha_2}, \cdots g_{i-1}^{\alpha_{i-1}}, g_i^{\nu_i}, g_{i+1}^{\nu_{i+1}}, \ldots) \leq \sup C_g^i \leq ge_{i-1}.$$

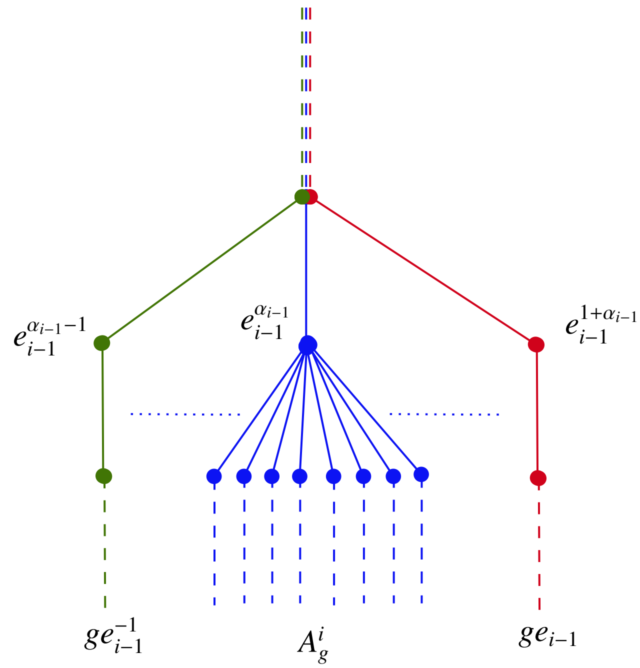


FIGURE II.7. Representation in the tree of the upper and lower bound of A_g^i

The assumption that $\sup C_g^i = (g_j^{\beta_j})_{j\in\mathbb{N}} \in \mathcal{G}$ leads to a contradiction, because from the above inequality we deduce that

$$\beta_j = \alpha_j \qquad \forall j < i-1$$

and

$$\beta_{i-1} = \alpha_{i-1} \quad or \quad \beta_{i-1} = 1 + \alpha_{i-1}.$$

Therefore,

a) If $\beta_{i-1} = \alpha_{i-1}$ then the element $e_1^{\alpha_1} \cdots e_{i-1}^{\alpha_{i-1}} e_i^{\beta_i+1}$ is in C_g^i and is larger than $\sup C_g^i$.

b) If $\beta_{i-1} = 1 + \alpha_{i-1}$ then the element $e_1^{\alpha_1} \cdots e_{i-1}^{1+\alpha_{i-1}} e_i^{\beta_i-1}$ is an upper bound of C_g^i and is less than $\sup C_g^i$.

For the case of $\inf C_g^i$, the proof has the same structure.

(iv) By definition, $A_g^i \subset C_g^i$ and it is clear that $\sup A_g^i \leq \sup C_g^i$. In order to show that $\sup A_g^i \geq \sup C_g^i$ we note that for $h \in C_g^i \setminus A_g^i$ we can always determine an element $f \in A_g^i$ such that $h < f$. Indeed, let $g = (g_i)_{i\in\mathbb{N}}^{\alpha_i}$. If $h = (g_j^{\beta_j})_{j\in\mathbb{N}} \in C_g^i$, then $f = ge_i^{\beta_j-\alpha_j+1}$ is an element A_g^i larger than h. Thus $\sup C_g^i = \sup A_g^i$.

(v) We suppose that $i < k$, in this case, the elements $h_1 = ge_i$ and $h_2 = ge_i^2$ belong to A_g^i and they satisfy $f < h_1 < h_2$ for all $f \in A_g^k$. Therefore, $\sup A_g^k < \sup A_g^i$.

□

II.2.2. The convex subgroups of $\mathcal{G}$. The following proposition gives us a description of the proper convex subgroups of $\mathcal{G}$.

Proposition II.2.2.
For each $i \in \mathbb{N}$ with $i > 1$, let

$$C_i = \left\{ g = \left(g_j^{\alpha_j}\right)_{j\in\mathbb{N}} : \alpha_j \in \mathbb{Z} \wedge \forall j < i, \alpha_j = 0 \right\}.$$

Then every set C_i is a proper convex subgroup of $\mathcal{G}$ and there are no others; thus $\mathcal{G}$ has infinite rank.

Proof.
In fact, the first $i-1$ components of the elements in C_i, are equal to 1. Let $f, h \in C_i$ with $f < h$ and let $g = (g_j^{\alpha_j})_{j\in\mathbb{N}} \in \mathcal{G}$ such that $f < g < h$. If there exists $\alpha_j \neq 0$ with $j < i$ then $g < f$ or $h < g$, which would contradict the definition of g. Then, necessarily, $\alpha_j = 0$ for all $j < i$. Therefore $g \in C_i$.

On the other hand, the collection of convex subgroups of $\mathcal{G}$ is a totally ordered set. Clearly we have that $C_i \supsetneq C_j$ when $i < j$. Suppose that H is a proper convex subgroup of $\mathcal{G}$, $H \neq C_i$ for all $i > 1$, then there exists $1 < j \in \mathbb{N}$ such that $C_{j+1} \subsetneq H \subsetneq C_j$. The strict inclusion above implies the existence of an element $h = (g_i^{\alpha_i})_{i\in\mathbb{N}} \in H$ such that $\alpha_i = 0$ for all $i < j$ and $\alpha_j > 0$. In addition H is convex, thus every element $f = (g_i^{\gamma_i})_{i\in\mathbb{N}} \in G$ such that $\gamma_i = 0$ for all $i < j$ and $-n\alpha_j < \gamma_j < n\alpha_j$ for some $n \in \mathbb{N}$ also belongs to H. Then $H = C_j$, and we obtain a contradiction to the strict inclusion above. □

Proposition II.2.3.
Consider the set $O_i = \left\{e_i^n : n \in \mathbb{Z}\right\}$. For any $g, h \in \mathcal{G}$ and $1 < i \in \mathbb{N}$ we have that

(1) $\sup A_g^i = g \cdot \sup O_i$ *and* $\inf A_g^i = g \cdot \inf O_i$.
(2) $\sup O_i = \inf(e_{i-1} O_i)$ *and* $\inf O_i = \sup\left(e_{i-1}^{-1} O_i\right)$.
(3) If $j < k$ then
  (i) $\sup O_k \bullet \sup O_j = \sup O_j$ *and* $\inf O_k \star \inf O_j = \inf O_j$,
  (ii) $\sup O_j \bullet \inf O_j = \inf O_j$ *and* $\sup O_j \star \inf O_j = \sup O_j$
  (iii) $\sup A_g^k \bullet \sup A_h^j = \sup A_{gh}^j$
  (iv) $\sup A_g^k \star \sup A_h^j = \sup A_{gh}^j$
(4) $Stab(\sup A_g^i) = C_{1_\mathcal{G}}^i$

Proof.
(1) $\sup A_g^i = \sup\left\{g e_i^n : n \in \mathbb{Z}\right\} = g \cdot \sup\left\{e_i^n : n \in \mathbb{Z}\right\} = g \cdot \sup O_i$. In the same way we prove $\inf A_g^i = g \cdot \inf O_i$.

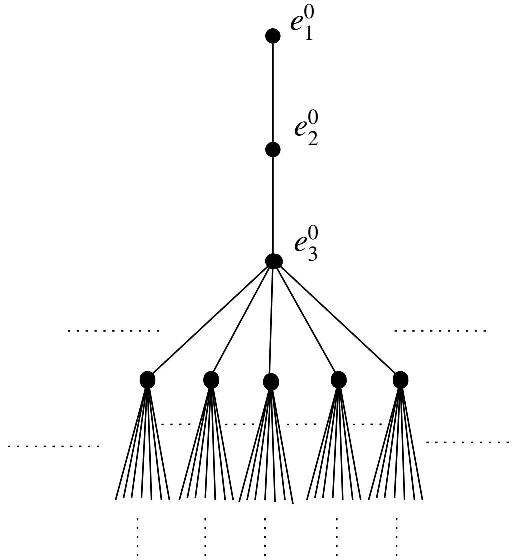


FIGURE II.8. Representation of the proper convex subgroup C_3

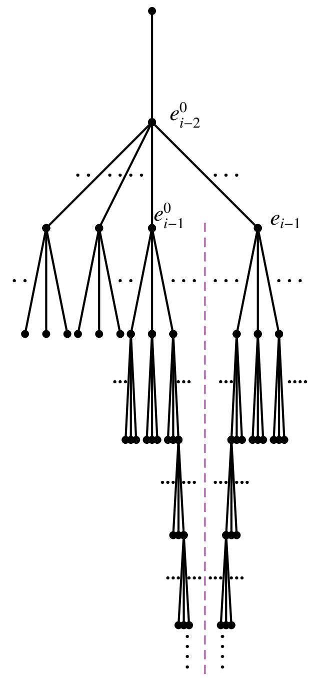
(2) If $\sup O_i < \inf(e_{i-1}O_i)$ then there exists $h = (g_j^{\gamma_j})_{j\in\mathbb{N}} \in \mathcal{G}$ such that $\sup O_i < h < \inf(e_{i-1}O_i)$. This last condition implies that $\gamma_j = 0$ for all $j < i-1$. As $h > \sup O_i$ necessarily $\gamma_{i-1} = 1$ but this is impossible, because $h < \inf(e_{i-1}O_i)$.
(3) We know that the dot and star multiplications are associative and if $g \in \mathcal{G}$, $x \in \mathcal{G}^\#$ then $g \bullet x = g \star x$.
    (i) First, note that for $i < k$, $e_j^{q-1} < e_k^p e_j^q < e_j^{q+1}$. Now,

$$\begin{aligned}\sup O_k \bullet \sup O_j &= \sup_{\mathcal{G}^\#}\left\{uv : u, v \in \mathcal{G}, u < \sup O_k, v < \sup O_j\right\}\\ &= \sup_{\mathcal{G}^\#}\left\{e_k^p e_j^q : p, q \in \mathbb{N}\right\} = \sup_{\mathcal{G}^\#}\left\{e_j^q : q \in \mathbb{N}\right\} = \sup O_j.\end{aligned}$$

$$\begin{aligned}\inf O_k \star \inf O_j &= \inf_{\mathcal{G}^\#}\left\{uv : u, v \in \mathcal{G}, u > \inf O_k, v > \inf O_j\right\}\\ &= \inf_{\mathcal{G}^\#}\left\{e_k^p e_j^q : p, q \in \mathbb{N}\right\} = \inf_{\mathcal{G}^\#}\left\{e_j^q : q \in \mathbb{N}\right\} = \inf O_j.\end{aligned}$$

    (ii)

$$\begin{aligned}\sup O_j \bullet \inf O_j &= \sup O_j \bullet \sup(O_j e_{j-1}^{-1})\\ &= e_{j-1}^{-1} \sup O_j \bullet \sup(O_j)\\ &= e_{j-1}^{-1} \sup O_j\\ &= \inf O_j.\end{aligned}$$



$\sup O_i = \inf\,(e_{i-1}O_i)$

$$\begin{aligned}
\sup O_j \star \inf O_j &= \inf(O_j e_{j-1}) \star \inf O_j \\
&= e_{j-1} \inf O_j \star \inf(O_j) \\
&= e_{j-1} \inf O_j \\
&= \sup O_j.
\end{aligned}$$

(iii)

$$\begin{aligned}
\sup A^k_g \bullet \sup A^j_h &= (g \cdot \sup O_i) \bullet (h \cdot \sup O_k) \\
&= gh\left(\sup O_k \bullet \sup O_j\right) \\
&= gh \cdot \sup O_j \\
&= \sup A^j_{gh}.
\end{aligned}$$

(iv)

$$\begin{aligned}
\sup A^k_g \star \sup A^j_h &= (g \cdot \sup O_k) \star \left(h \cdot \sup O_j\right) \\
&= gh \cdot \left[\inf\,(e_{k-1}O_k) \star \inf\left(e_{j-1}O_j\right)\right] \\
&= ghe_{k-1}e_{j-1} \cdot \left[\inf\,(O_k) \star \inf\left(O_j\right)\right] \\
&= ghe_{k-1}e_{j-1} \inf\left(O_j\right) \\
&= ghe_{k-1} \sup\left(O_j\right)
\end{aligned}$$

$$= \; gh \sup\left(O_j\right)$$
$$= \; \sup A^j_{gh}.$$

(4) $Stab\left(\sup A^i_g\right) = Stab\left(g \cdot \sup O_i\right) = Stab\left(\sup O_i\right) = Stab\left(\sup C^i_{1_{\mathcal{G}}}\right) = C^i_{1_{\mathcal{G}}}$

□

The rank of the totally ordered group $\mathcal{G}$ is ω. Thus $\mathcal{G}$ is not complete. In the next theorem we will characterize its completion $\mathcal{G}^\#$.

Theorem II.2.1.

$$\mathcal{G}^\# = \mathcal{G} \cup \bigcup_{1<i\in\mathbb{N}} \mathcal{G} \sup O_i.$$

Proof.
The existence of $u \in \mathcal{G}^\# \setminus \mathcal{G}$ that does not belong to any orbit $\mathcal{G} \sup O_i$ with $i > 1$ will lead to a contradiction. Indeed, we know that

$$u = \sup_{\mathcal{G}^\#} \left\{h = \left(g_i^{\alpha_i(h)}\right)_{i\in\mathbb{N}} \in \mathcal{G} : h < u\right\}$$

We denote by $B^1_u = \left\{h = \left(g_i^{\alpha_i(h)}\right)_{i\in\mathbb{N}} \in G : h < u\right\}$. Necessarily there is an $m_1 \in \mathbb{N}$ such that

$$m_1 = \max\left\{\alpha_1(h) : h \in B^1_u\right\}$$

since, otherwise, B^1_u is not a bounded set. Now, denote by

$$B^2_u = \left\{h \in B^1_u : \alpha_1(h) = m_1\right\}$$

We have that $m_2 = \max\left\{\alpha_2(h) : h \in B^2_u\right\}$ must exist. If not, $u = \sup(e_1^{m_1} O_2)$ which is impossible by the definition of u. Denote now, for each $k \in \mathbb{N}$,

$$B^k_u = \left\{h \in B^{k-1}_u : \alpha_{k-1}(h) = m_{k-1}\right\}$$

and we will prove by induction that $m_k = \max\left\{\alpha_k(h) : h \in B^k_u\right\}$ does exist for all $k \in \mathbb{N}$.

We have already proved the existence of m_1 and m_2. Now, suppose that $m_k = \max\left\{\alpha_k(h) : h \in B^k_u\right\}$ exists and so

$$B^{k+1}_u = \left\{h \in B^k_u : \alpha_k(h) = m_k\right\}.$$

If we suppose that $m_{k+1} = \max\left\{\alpha_{k+1}(h) : h \in B^{k+1}_u\right\}$ does not exist we are led to

$$u = e_1^{m_1} e_2^{m_2} \cdots e_k^{m_k} \sup(O_{k+1}),$$

which contradicts the definition of u. We conclude that there is m_k, as defined above, for all $k \in \mathbb{N}$.

Thus,

$$u = \sup_{G^{\#}} \left\{ h = \left(g_i^{\alpha_i(h)} \right)_{i\in\mathbb{N}} \in G : h < u \right\} = (g_i^{m_i})_{i\in\mathbb{N}}$$

and this contradicts the definition of u. □

Proposition II.2.4. *If $1 < i \in \mathbb{N}$ then $\mathcal{G}/C^i_{1_{\mathcal{G}}}$ is quasidiscrete.*

Proof. Note that,

$$\begin{aligned}
\sup(C^i_{1_{\mathcal{G}}}) \star \sup(C^i_{1_{\mathcal{G}}}) &= \sup(O_i) \star \sup(O_i) \\
&= \inf(O_i e_{i-1}) \star \inf(O_i e_{i-1}) \\
&= e^2_{i-1} \inf(O_i) \\
&= e_{i-1} \sup(O_i) \\
&= e_{i-1} \sup(C^i_{1_{\mathcal{G}}}).
\end{aligned}$$

$$\begin{aligned}
\inf(C^i_{1_{\mathcal{G}}}) \bullet \inf(C^i_{1_{\mathcal{G}}}) = &= \inf(O_i) \bullet \inf(O_i) \\
&= \sup(O_i e^{-1}_{i-1}) \bullet \sup(O_i e^{-1}_{i-1}) \\
&= e^{-2}_{i-1} \sup(O_i) \\
&= e^{-1}_{i-1} \inf(O_i) \\
&= e^{-1}_{i-1} \inf(C^i_{1_{\mathcal{G}}}).
\end{aligned}$$

By Proposition I.2.5, it is known that $\mathcal{G}/C^i_{1_{\mathcal{G}}}$ is quasidiscrete with $\pi(e_{i-1}) = \min\{h \in \mathcal{G}/C^i_{1_{\mathcal{G}}} : h > 1\}$, where $\pi : \mathcal{G} \to \mathcal{G}/C^i_{1_{\mathcal{G}}}$ is the canonical map.

□

II.3. Example: Levi-Civita Field

We start by stating the main definitions and properties of Levi-Civita field.

A subset M of the rational numbers $\mathbb{Q}$ is called left-finite if for every $r \in \mathbb{Q}$ there are only finitely many elements of M that are smaller than r. The set of all left-finite subsets of $\mathbb{Q}$ will be denoted by $\mathcal{F}$.

Let $M \in \mathcal{F}$. If $M \neq \emptyset$, the elements of M can be arranged in ascending order; and there exists a minimum of M. If M is infinite, its elements form a strictly monotonic sequence that is divergent.

Also, we have for $M, N \in \mathcal{F}$

(1) $M \cup N \in \mathcal{F}$, $M \cap N \in \mathcal{F}$ and if $X \subset M$ then $X \in \mathcal{F}$,
(2) $M + N = \{a + b : a \in M, b \in N\} \in \mathcal{F}$, and for every $c \in M + N$, there are only finitely many pairs $(a, b) \in M \times N$ such that $c = a + b$.

Definition II.3.1.
Consider the set $\mathcal{R}$ of all real-valued functions on $\mathbb{Q}$ that are nonzero only on a left-finite

set, that is, they have left-finite support

$$\mathcal{R} := \{f : \mathbb{Q} \to \mathbb{R} : supp(f) \in \mathcal{F}\}.$$

We define two operations for the elements in $\mathcal{R}$. Let $f, g \in \mathcal{R}$ and $q \in \mathbb{Q}$:

(i) *the addition on $\mathcal{R}$ is componentwise*

$$(f + g)[q] = f[q] + g[q]$$

(ii) *and multiplication is defined as follows*

$$(f \cdot g)[q] = \sum_{q_1+q_2=q} f[q_1] \cdot g[q_2]$$

We have that $\mathbb{R}$ can be embedded into $\mathcal{R}$ via the map Π. Let $x \in \mathbb{R}$, the map $\Pi : \mathbb{R} \to \mathcal{R}$ is defined by

$$\Pi(x)[q] = \begin{cases} x & \text{, if } q = 0 \\ 0 & \text{, else} \end{cases}$$

Π is injective, $\Pi(x + y) = \Pi(x) + \Pi(x)$ and $\Pi(x \cdot y) = \Pi(x) \cdot \Pi(x)$. This embedding is not surjective, note that if $x \in \mathbb{R} \setminus \{0\}$, $supp(\Pi(x)) = \{0\}$.

Theorem II.3.1. *(*[**2**]*, Theorem 2.3)*
$(\mathcal{R}, +, \cdot)$ is a field.

The field $(\mathcal{R}, +, \cdot)$ is called the Levi-Civita field ([**2**] and [**17**] contain interesting results with respect to this field). From now on, its elements will be denoted by the letters $x, y, z, ...$ (instead of $f, g, h...$). Also we denote the identity element in $\mathcal{R}$ by 1, so

$$1[q] = \begin{cases} 1 & \text{, if } q = 0 \\ 0 & \text{, else} \end{cases}$$

For $x \in \mathcal{R}$ with $x \neq 0$, we denote by $\lambda(x) = \min(supp(x))$ which exists because of left-finiteness of $supp(x)$ and we define $\lambda(0) = +\infty$.

To introduce an order structure to $\mathcal{R}$, we consider the set $\mathcal{R}^+$ of all nonvanishing elements $x \in \mathcal{R}$ that satisfy $x[\lambda(x)] > 0$. This is the cone of positivity in the Levi-Civita Field.

$$\mathcal{R}^+ = \{x \in \mathcal{R} \setminus \{0\} : x[\lambda(x)] > 0\}.$$

The basic properties of $\mathcal{R}^+$ are:

Lemma II.3.1. *(*[**2**]*, Lemma 3.1)*

(i) *$\mathcal{R}^+ \cap (-\mathcal{R}^+) = \emptyset$, $\mathcal{R}^+ \cap \{0\} = \emptyset$ and $\mathcal{R}^+ \cup \{0\} \cup (-\mathcal{R}^+) = \mathcal{R}$*

(ii) *If $x, y \in \mathbb{R}^+$, then $x + y \in \mathbb{R}^+$ and $xy \in \mathbb{R}^+$.*

Now, we define an order in $\mathcal{R}$.

Definition II.3.2.
Let $x, y \in \mathcal{R}$ be given. We say that $y > x$ if $x \neq y$ and $(y-x) \in \mathcal{R}^+$; and we say $y \geq x$ if $y = x$ or $y > x$. Also, we say $y < x$ if $x > y$ and $y \leq x$ if $x \geq y$.

Notice that with this definition we have, for $x \neq y$,

$$y > x \Leftrightarrow (y-x)[\lambda(y-x)] > 0$$

Theorem II.3.2. *(*[**2**]*, Theorem 3.1)*
With the relation $\geq$, $(\mathcal{R}, +, \cdot)$ becomes a totally ordered field.

The order is compatible with the algebraic structure of $\mathcal{R}$, that is, for any x, y, z, we have: $x > y \Rightarrow x + z > y + z$; and if $z > 0$, we have $x > y \Rightarrow x \cdot z > y \cdot z$.

II.3.1. Levi-Civita Multiplicative Group $(\mathcal{R}^+, \cdot)$. Later on, we will see that the multiplicative group $(\mathcal{R}^+, \cdot)$ has infinite rank. Our aim here is the description of its convex subgroups. In fact, if we consider the order given by the inclusion, we will show that

(i) the set
$$\mathcal{L} = \{x \in \mathcal{R}^+ : \lambda(x) = 0\}$$
is the largest convex subgroup of $(\mathcal{R}^+, \cdot)$

(ii) the largest proper convex subgroup contained in $\mathcal{L}$ is
$$\mathcal{L}^0 = \{x \in \mathcal{R}^+ : \lambda(x) = 0, x[0] = 1\}$$

(ii) for each $r \in \mathbb{Q}^+$, the sets $\mathcal{L}^0_r = \{x \in \mathcal{R}^+ : \lambda(x) = 0, x[0] = 1, \lambda_1(x) \geq r\}$ are convex subgroups of $(\mathcal{R}^+, \cdot)$ where
$$\lambda_1(x) := \begin{cases} \min\left(supp(x) \setminus \{\lambda(x)\}\right) & \text{if } supp(x) \neq \{\lambda(x)\} \\ +\infty & \text{else} \end{cases}$$

Before we characterize the positive cone $P = \{x \in \mathcal{R}^+ : x > 1\}$ we will show by an example different types of elements of P.

Example 10.
We consider the following numbers:

$$x[q] = \begin{cases} 1 & , q = -2 \\ -1 & , q = -1 \\ 2 & , q = 1 \\ 0 & , \text{otherwise} \end{cases} \qquad y[q] = \begin{cases} 2 & , q = 0 \\ -1 & , q = 1 \\ 3 & , q = 2 \\ 0 & , \text{otherwise} \end{cases} \qquad z[q] = \begin{cases} 1 & , q = 0 \\ 3 & , q = 2 \\ -1 & , q = 5 \\ 0 & , \text{otherwise} \end{cases}$$

See Figure II.9 for a graphic representation of these elements. They are in the positive cone P of $(\mathcal{R}^+, \cdot)$. Indeed,

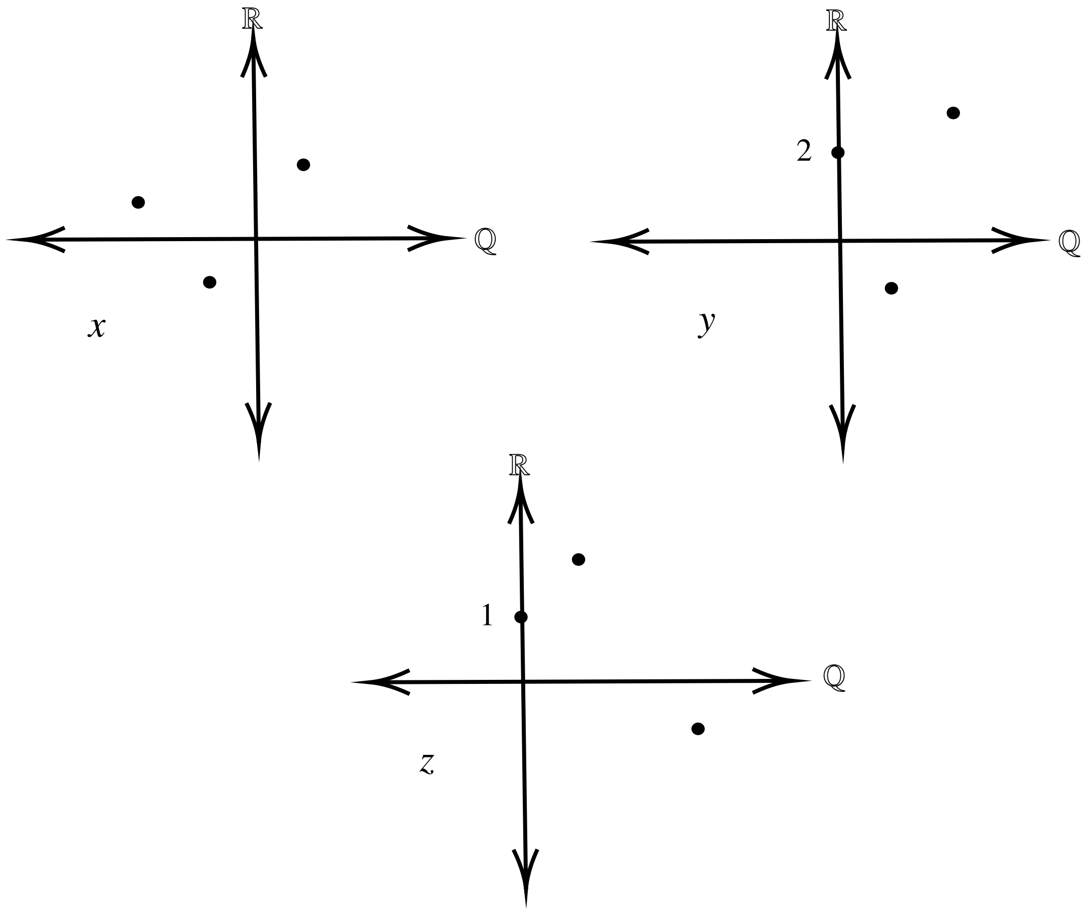


FIGURE II.9. The numbers $x, y, z \in P$

$$\begin{array}{lll} \lambda(x-1) = -2 & \textit{and} & (x-1)[-2] = 1 > 0 \\ \lambda(y-1) = 0 & \textit{and} & (y-1)[0] = 1 > 0 \\ \lambda(z-1) = 2 & \textit{and} & (z-1)[2] = 3 > 0 \end{array}$$

These three numbers allow us to visualize three subsets of the positive cone P.

(i) The subset P_1 of elements such that the minimum of the support is negative ($x \in P_1$).
(ii) The subset P_2 of elements such that the minimum of the support is 0 and whose value at 0 is greater than 1 ($y \in P_2$).
(iii) The subset P_3 of elements such that the minimum of the support is 0, whose value at 0 is 1 and whose value at the next support point is greater than 0 ($z \in P_3$).

Lemma II.3.2.
Let

$$\lambda(x) := \begin{cases} \min(supp(x)) & \text{if } x \neq 0 \\ +\infty & \text{else} \end{cases}$$

$$\lambda_1(x) := \begin{cases} \min(supp(x) \setminus \{\lambda(x)\}) & \text{if } supp(x) \neq \{\lambda(x)\} \\ +\infty & \text{else} \end{cases}$$

and P_1, P_2 and P_3 the following subsets of P

$$\begin{aligned} P_1 &= \{x \in \mathcal{R}^+ : \lambda(x) < 0\} \\ P_2 &= \{x \in \mathcal{R}^+ : \lambda(x) = 0 \wedge x[0] > 1\} \\ P_3 &= \{x \in \mathcal{R}^+ : \lambda(x) = 0 \wedge x[0] = 1 \wedge x[\lambda_1(x)] > 0)\}. \end{aligned}$$

Then $P = P_1 \cup P_2 \cup P_3$.

Proof.
If $x \in (P_1 \cup P_2 \cup P_3)$ then $(x-1)[(\lambda(x-1)] > 0$ because $1[q] = \begin{cases} 1 & \text{if } q = 0 \\ 0 & \text{else} \end{cases}$ and

- if $x \in P_1$, then $\lambda(x-1) < 0$ and $(x-1)[\lambda(x-1)] = x[\lambda(x)] > 0$
- if $x \in P_2$, then $\lambda(x-1) = 0$ but $(x-1)[0] = x[0] - 1 > 0$
- if $x \in P_3$, then $\lambda(x-1) > 0$ and $(x-1)[\lambda(x-1)] = x[\lambda_1(x)] > 0$.

Therefore, $P_1 \cup P_2 \cup P_3 \subset P$. So, we only need to prove that $P \subset P_1 \cup P_2 \cup P_3$.

If $x \in P$ then $x[\lambda(x)] > 0$ and $\lambda(x) \leq 0$, since if we suppose $\lambda(x) > 0$ then $(1-x)[\lambda(1-x)] = (1-x)[0] = 1$, therefore $1 > x$ and $x \notin P$, a contradiction. If $x \in P$ and $x \notin (P_2 \cup P_3)$ then

(i) If $\lambda(x) = 0 \wedge x[0] < 1$ then $x < 1$ and hence $x \notin P$, a contradiction.
(ii) If $\lambda(x) = 0 \wedge x[0] = 1 \wedge x[\lambda_1(x)] < 0$ then $(x-1)[\lambda(x-1)] = (x)[\lambda_1(x)] < 0$; it follows that $x < 1$ and hence $x \notin P$, a contradiction.
(iii) If $\lambda(x) < 0$, the only option is $x[\lambda(x)] > 0$ and therefore $x \in P_1$.

□

Notice that $P \cap P^{-1} = \emptyset$, $P \cap \{1\} = \emptyset$, $P \cup \{1\} \cup P^{-1} = \mathcal{R}^+$ and $x, y \in P \Rightarrow x \cdot y \in P$. Also, the order defined on $\mathcal{R}$ induces an order in $(\mathcal{R}^+, \cdot)$. Thus

$$y > x \Leftrightarrow yx^{-1} \in P \Leftrightarrow yx^{-1} \in (P_1 \cup P_2 \cup P_3)$$

II.3.2. The convex subgroups of $(\mathcal{R}^+, \cdot)$. In order to determine the rank of $(\mathcal{R}^+, \cdot)$ we will study the following subsets of $\mathcal{R}^+$

$$\begin{aligned} \mathcal{L} &= \{x \in \mathcal{R}^+ : \lambda(x) = 0\} \\ \mathcal{L}^0 &= \{x \in \mathcal{R}^+ : \lambda(x) = 0, x[0] = 1\} \\ \mathcal{L}^0_r &= \{x \in \mathcal{R}^+ : \lambda(x) = 0, x[0] = 1, \lambda_1(x) \geq r\} \end{aligned}$$

where $r \in \mathbb{Q}^+$ and for all $x \in \mathcal{R}^+$ $\lambda(x)$ and $\lambda_1(x)$ are defined as in Lema II.3.2. We have for all $r \in \mathbb{Q}^+$, $\mathcal{L}_r^0 \subsetneq \mathcal{L}^0 \subsetneq \mathcal{L}$. We will prove that these sets belong to a decreasing chain of convex subgroups of $\mathcal{R}^+$.

Lemma II.3.3.
With the order given by inclusion, the set

$$\mathcal{L} := \{x \in \mathcal{R}^+ : \lambda(x) = 0\}$$

is the largest convex subgroup of $(\mathcal{R}^+, \cdot)$.

Proof.
Note that $1 \in \mathcal{L}$. Firstly, we must prove that $\mathcal{L}$ is closed under multiplication. Suppose $x, y \in \mathcal{L}$; then $\lambda(x) = \lambda(y) = 0$ and $\lambda(x \cdot y) = 0$, because $0 \leq q_1 + q_2$ with $q_1 \in supp(x)$, $q_2 \in supp(y)$ and $(xy)[0] = x[0] \cdot y[0] > 0$. Now, let $x \in \mathcal{L}$, we prove that $x^{-1} \in \mathcal{L}$. Indeed, we know that $0 = \lambda(1) = \lambda(xx^{-1}) = \lambda(x) + \lambda(x^{-1}) = \lambda(x^{-1})$, $x[0] > 0$ and $1 = (xx^{-1})[0] = x[0] \cdot x^{-1}[0]$ therefore $x^{-1}[0] = \dfrac{1}{x[0]} > 0$ and so $x^{-1} \in \mathcal{L}$. The other group axioms are inherited from $(\mathcal{R}^+, \cdot)$.

Next, let $x, y \in \mathcal{L}$ and $u \in \mathcal{R}^+$ such that $x < u < y$. This implies that $\lambda(u) = 0$ because if $\lambda(u) > 0$ then $u < x$, and if $\lambda(u) < 0$ then $u > y$. This shows that $\mathcal{L}$ is a convex subgroup.

Finally, let $\mathcal{M}$ be a proper convex subgroup of $(\mathcal{R}^+, \cdot)$. If $x \in \mathcal{M} \setminus \mathcal{L}$ then, without loss of generality, we may assume that $\lambda(x) < 0$. Thus, we can always find $n \in \mathbb{N}$ such that $\lambda(x) < -\frac{1}{n}$ and so the element $d^{-\frac{1}{n}}$, given by

$$d^{-\frac{1}{n}}[q] = \begin{cases} 1 & q = -\frac{1}{n} \\ 0 & q \neq -\frac{1}{n} \end{cases}$$

is in $\mathcal{R}^+$ and $1 < d^{-\frac{1}{n}} < x$.

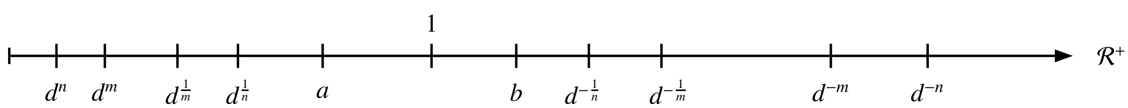


Figure II.10. $a, b \in \mathbb{R}$ with $a < 1 < b$; $n, m \in \mathbb{N}$ with $m < n$

Then, by the convexity of the subgroup $\mathcal{M}$ we have that $d^{-1} \in \mathcal{M}$. Moreover, for any $y \in \mathcal{R}^+$ we can always find a $m \in \mathbb{N}$ such that $d^m < y < d^{-m}$ and hence $y \in \mathcal{M}$, which contradicts that $\mathcal{M}$ is a proper subgroup of $\mathcal{R}^+$.

□

Lemma II.3.4.
The set

$$\mathcal{L}^0 = \{x \in \mathcal{R}^+ : \lambda(x) = 0, x[0] = 1\}$$

is a convex subgroup of $(\mathcal{R}^+, \cdot)$.

Proof.

Notice that if $x \in \mathcal{L}^0$ then $\lambda(x^{-1}) = 0$ and $x^{-1}[0] = x[0]x^{-1}[0] = 1[0] = 1$ and therefore $x^{-1} \in \mathcal{L}^0$. Let $x, y \in \mathcal{L}^0$ and $u \in \mathcal{R}^+$ such that $x < u < y$. Then

(i) $\lambda(u) = 0$: If $\lambda(u) > 0$ then $u < x$, and if $\lambda(u) < 0$ then $u > y$.

(ii) $u[0] = 1$: If $u[0] > 1$ then $u > y$, and if $u[0] < 1$ then $u < x$.

Thus $u \in \mathcal{L}^0$. □

Lemma II.3.5.
The set

$$\mathcal{L}^0_r = \{x \in \mathcal{R}^+ : \lambda(x) = 0, x[0] = 1, \lambda_1(x) \geq r\}$$

is a convex subgroup of $(\mathcal{R}^+, \cdot)$ *for each* $r \in \mathbb{Q}^+$.

Proof.

Just as before, we can prove that if $x \in \mathcal{L}^0_r$ then $\lambda(x^{-1}) = 0$ and $x^{-1}[0] = 1$.

Additionally, $\lambda_1(x^{-1}) \geq r$, since if $\lambda_1(x^{-1}) < r$ then $(x \cdot x^{-1})[\lambda_1(x^{-1})] = x[0] \cdot x^{-1}[\lambda_1(x^{-1})] = x^{-1}[\lambda_1(x^{-1})] \neq 0$ but on the other hand $(x \cdot x^{-1})[\lambda_1(x^{-1})] = 1[\lambda_1(x^{-1})] = 0$, a contradiction. We conclude that $x^{-1} \in \mathcal{L}^0_r$.

Finally, we show that $\mathcal{L}^0_r$ is a convex subgroup. Let $x, y \in \mathcal{L}^0_r$ and $u \in \mathcal{R}^+$ such that $x < u < y$. As in the proof of the previous Lemma, we have that $\lambda(u) = 0$ and $u[0] = 1$. Also $\lambda_1(u) \geq r$, because assuming that $\lambda_1(u) < r$ leads to a contradiction: If $u[\lambda_1(u)] < 0$ then $u < x$ and if $u[\lambda_1(u)] > 0$ then $u > y$.

Thus $u \in \mathcal{L}^0_r$ and therefore $\mathcal{L}^0_r$ is a convex group for all $r \in \mathbb{Q}^+$. □

Corollary II.3.1. $(\mathcal{R}^+, \cdot)$ *has infinite rank.*

Lemma II.3.6.
There is a jump in $(\mathcal{R}^+, \cdot)$*, i.e.* $\mathcal{R}^+$ *does not contain any convex subgroup between* $\mathcal{L}^0$ *and* $\mathcal{L}$*.*

Proof.

If $\overline{\mathcal{L}}$ were a convex subgroup of $(\mathcal{R}^+, \cdot)$ such that $\mathcal{L}^0 \subsetneq \overline{\mathcal{L}} \subseteq \mathcal{L}$, then there would be an element $x \in \overline{\mathcal{L}}$ such that $\lambda(x) = 0$, $x[0] = a \neq 1$ (if for all $x \in \overline{\mathcal{L}}$, $x[0] = 1$, then $\overline{\mathcal{L}} = \mathcal{L}^0$). But we know that $\overline{\mathcal{L}}$ is a convex subgroup and in this case $x^{-1}[0] = a^{-1}$. Without loss of generality, we may assume that $a > 1$. Thus, for any element $y \in \mathcal{L}$ such that $y[0] = b$ we can find an $n \in \mathbb{N}$ such that $a^{-n} < b < a^n$. Note that $x^n[0] = a^n$ and $x^{-n}[0] = a^{-n}$ and therefore $x^{-n} < y < x^n$, so by convexity of $\overline{\mathcal{L}}$ we have that $y \in \overline{\mathcal{L}}$. We conclude that $\overline{\mathcal{L}} = \mathcal{L}$. □

Lemma II.3.7.
For all $r \in \mathbb{Q}^+$*, the largest proper convex subgroup in* $\mathcal{L}^0_r$ *is* $(\mathcal{L}^0_r)^\star = \bigcup_{i>r} \mathcal{L}^0_i$

Proof.
If $\overline{\mathcal{L}}$ were a convex subgroup of $(\mathcal{R}^+, \cdot)$ such that

$$\bigcup_{i>r} \mathcal{L}_i^0 \subsetneq \overline{\mathcal{L}} \subseteq \mathcal{L}_r^0$$

then we can find an element $x \in \overline{\mathcal{L}}$ such that $x \notin \mathcal{L}_i^0$ for all $i > r$. This condition implies that $\lambda_1(x) = r$ otherwise $\lambda_1(x) > r$ and $x \in \mathcal{L}_{\lambda_1(x)}^0 \subset \bigcup_{i>r} \mathcal{L}_i^0$.

Now, we prove the inclusion $\mathcal{L}_r^0 \subseteq \overline{\mathcal{L}}$. Indeed, let $x \in \mathcal{L}_r^0$ and put $a := x[r] \in \mathbb{R}$; then for all $y \in \mathcal{L}_r^0$ with $y[r] = b$ we could find $n \in \mathbb{N}$ such that $-n|a| < b < n|a|$. Also we know that

$$\lambda_1(x^n) = \lambda_1(x^{-n}) = r, \quad x^n[r] = na \quad \text{and} \quad x^{-n}[r] = -na.$$

Note that if $a > 0$ then $x^{-n} < y < x^n$, and if $a < 0$ then $x^n < y < x^{-n}$. Therefore $y \in \overline{\mathcal{L}}$, because x^n and x^{-n} are elements of the convex subgroup $\overline{\mathcal{L}}$. Thus $\overline{\mathcal{L}} = \mathcal{L}_r^0$ and hence $(\mathcal{L}_r^0)^\star$ is the greatest convex subgroup contained in $\mathcal{L}_r^0$. □

These two examples above show the behavior of some totally ordered groups with a decreasing sequence of convex subgroups.

The results in the next section will complete the characterization of $(G^{\#})_0$.

II.4. Sufficient conditions for $G \subsetneq (G^{\#})_0$

Theorem II.4.1.
Let $(G, \cdot, \leq)$ be a totally ordered multiplicative group with $rank(G) > 1$. Let $G^{\#}$ be the Dedekind completion of G, and let $\alpha \in G^{\#} \setminus G$ and $C_\alpha := \{g \in G : g < \alpha\}$. If there exists a convex subgroup H of G such that $H \subseteq C_\alpha$ then $H \subseteq Stab(\alpha)$.

Proof.
Without loss of generality, we may assume that $\alpha > 1$. Since G is cofinal (and coinitial) in $G^{\#}$, we can always find $f \in G$ such that $\alpha < f$. Thus C_α is a bounded above subset of G (see Figure II.11).

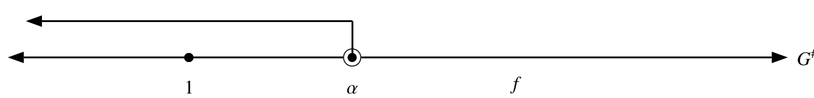


Figure II.11. Representation of C_α

Consider now the canonical morphism

$$\begin{aligned} \pi : \quad & G \to G/H \\ & g \longmapsto \overline{g} = gH. \end{aligned}$$

We know that π is increasing. We will prove that $\pi(C_\alpha)$ is a cut in G/H.

(i) $C_\alpha \neq \emptyset$, since G is coinitial in $G^{\#}$ we can find $u \in G$ such that $u < \alpha$ and therefore $\pi(C_\alpha) \neq \emptyset$.

(ii) On the other hand, if $\pi(u) < \pi(g)$ for any $g \in C_\alpha$ then $u \leq g$ (because π is increasing), and since C_α is a cut in G then $u \in C_\alpha$, therefore $\pi(u) \in \pi(C_\alpha)$.

Let $s = \sup_{(G/H)^\#} \pi(C_\alpha)$, where $(G/H)^\#$ is the Dedekind completion of the group G/H, and let $h \in H$ with $h > 1$. Then

$$h \cdot \alpha = h \cdot \sup_{G^\#} \{g \in G : g < \alpha\} = \sup_{G^\#} \{h \cdot g \in G : g \in C_\alpha\}$$

and since $h > 1$ then $\alpha \leq h \cdot \alpha$. Note that, for all $g \in C_\alpha$, $\pi(h \cdot g) = \pi(h) \cdot \pi(g) = \pi(g)$ and therefore

$$\sup_{(G/H)^\#} \pi(h \cdot C_\alpha) = \sup_{(G/H)^\#} \pi(C_\alpha) = s.$$

This implies that for all $g \in C_\alpha$ there is $u \in C_\alpha$ such that $hg \leq u$; thus, $hg \in C_\alpha$ and hence $h\alpha \leq \alpha$. We conclude that $h \cdot \alpha = \alpha$ and therefore $h \in Stab(\alpha)$. □

Corollary II.4.1.
If G contains a chain C of non trivial convex subgroups such that

$$\bigcap_{\Gamma \in C} \Gamma = \{1\} \ \textit{then} \ G = (G^\#)_0$$

Proof.
Under the same assumptions as in the previous theorem, let $\alpha \in (G^\#) \setminus G$. If G contains a decreasing chain of convex subgroups that converges to the trivial subgroup $\{1\}$, then there exists a convex subgroup $H \in C$ such that $H \subsetneq C_\alpha$ and therefore $H \subseteq Stab(\alpha)$. For instance, we can consider in the previous theorem $H = \bigcup_{\substack{\Gamma \in C \\ \alpha \notin \Gamma}} \Gamma$. □

Lemma II.4.1.
let $\alpha \in G^\# \setminus G$ and $C_\alpha := \{g \in G : g < \alpha\}$. Under the same assumptions as in the Theorem II.4.1, if C_α contains no convex subgroup except the trivial one then $G \subsetneq (G^\#)_0$.

Proof.
Let

$$H = \bigcap_{\Gamma_i \subseteq G} H_i$$

where for all i, Γ_i is a non trivial convex subgroup of G. Then $\alpha < sup(H)$, because otherwise $H \subseteq C_\alpha$, which contradicts our hypothesis. Therefore $\alpha \in H^\#$ and there are $h_1, h_2 \in H$ such that $h_1 < \alpha < h_2$.

Since $rank(H) = 1$, there exists an isomorphism from H to a multiplicative subgroup S of $\mathbb{R}^+$. S cannot be a discrete subgroup, i. e. $S \neq \langle g \rangle$ for all $g \in \mathbb{R}$, because in this case $H^\# = H$ and $\alpha \notin H$. It follows that S is a dense subgroup of $\mathcal{R}^+$, i. e. $H^\# = \mathbb{R}^+$ and therefore there exists $\alpha^{-1} \in \mathcal{R}^+$ such that $\alpha \cdot \alpha^{-1} = 1$ and $\alpha \in (G^\#)_0$. □

Corollary II.4.2.
$(\mathcal{R}^+)^{\#}_0 = \mathcal{R}^+$ and $(\mathcal{G}^{\#})_0 = \mathcal{G}$.

Proof.
Both of these totally ordered groups contain an infinite decreasing chain of convex subgroups. Indeed, in $(\mathcal{R}^+,\cdot)$, for all $p, q \in \mathbb{Q}^+$ with $p < q$ we have that

$$\mathcal{R}^+ \supsetneq \mathcal{L} \supsetneq \mathcal{L}^0_1 \supsetneq \left(\mathcal{L}^0_1\right)^\star \supsetneq \cdots \supsetneq \mathcal{L}^0_p \supsetneq \left(\mathcal{L}^0_p\right)^\star \supsetneq \cdots \supsetneq \mathcal{L}^0_q \supsetneq \left(\mathcal{L}^0_q\right)^\star \supsetneq \cdots \supsetneq \{1\}.$$

On the other hand, for the direct sum $(\mathcal{G},\cdot)$, for all $i, j \in \mathbb{N}$ with $i < j$ we have that $C_i \supsetneq C_j$; therefore the group $\mathcal{G}$ contains a decreasing sequence of convex subgroups such that

$$\mathcal{G} \supsetneq C_1 \supsetneq C_2 \supsetneq \cdots \supsetneq C_i \supsetneq C_{i+1} \supsetneq \cdots \supsetneq C_j \supsetneq \cdots \supsetneq \{1\}.$$

From Corollary II.4.1, we deduce that $(\mathcal{R}^+)^{\#}_0 = \mathcal{R}^+$ and $(\mathcal{G}^{\#})_0 = \mathcal{G}$. □

Corollary II.4.3.
Let H and Γ be totally ordered groups, with rank$(\Gamma) = 1$. Let $G = H \times \Gamma$ be the direct sum of H and Γ, with componentwise multiplication and lexicographic ordering. We have two cases for $(G^{\#})_0$:

- *If Γ is a cyclic subgroup of $(\mathbb{R}^+,\cdot)$ then $(G^{\#})_0 = G$.*
- *If Γ is a dense subgroup of $(\mathbb{R}^+,\cdot)$ then $(G^{\#})_0 = H \times \mathbb{R}^+$.*

Proof.
$G = H \times \Gamma$ is a totally ordered group with a first convex subgroup $\{1\} \times \Gamma$. If Γ is a cyclic subgroup of $(\mathbb{R}^+,\cdot)$ then, by Corollary II.1.1, we have that $(G^{\#})_0 = G$. On the other hand, if Γ is a dense subgroup of $(\mathbb{R}^+,\cdot)$ then, by Theorem II.4.1, we have that $(G^{\#})_0 = H \times \mathbb{R}^+$. □

We can summarize the description of $(G^{\#})_0$ as follows. Let G be a totally ordered group.

(i) If G contains a chain C of non trivial convex subgroups such that

$$\bigcap_{\Gamma \in C} \Gamma = \{1\} \text{ then } G = (G^{\#})_0$$

(ii) If the first non trivial convex subgroup of G is isomorphic to a proper dense subgroup of $\mathbb{R}^+$ then $G^{\#}$ contains a group larger than G; otherwise G is the largest group contained in $G^{\#}$.

CHAPTER III

Order in $M(X, G^{\#})$

Let G a totally ordered group and $G^{\#}$ its Dedekind completion. Let X be a G-module and $M(X, G^{\#})$ be the set of all the G-module maps from X to $G^{\#}$ such that

$$M(X, G^{\#}) = \left\{\varphi : X \to G^{\#} : \varphi \text{ is increasing and } \forall g \in G,\ \varphi(gx) = g\varphi(x)\right\}$$

We consider the natural ordering on $M(X, G^{\#})$ given by

$$\varphi_1 \leq \varphi_2 \Leftrightarrow \varphi_1(x) \leq \varphi_2(x), \quad \text{for all } x \in X$$

In this chapter our aim is to extend the results in [**14**], where the authors determined all G-module maps $G^{\#} \to G^{\#}$. We will study the set $M(X, G^{\#})$, where X can be any G-module and prove that $M(X, G^{\#})$ is a totally ordered set; even more, it is a G-module. For this end, we start in Section III.1 by studying the order of two families of morphisms contained in $M(X, G^{\#})$.

$$M(X, G^{\#})_{\sup} = \left\{f_{x_0} \in M(X, G^{\#}) : f_{x_0}(x) = \sup_{G^{\#}}\{g \in G : gx_0 \leq x\}, x_0 \in X\right\}$$

$$M(X, G^{\#})_{\inf} = \left\{h_{x_0} \in M(X, G^{\#}) : h_{x_0}(x) = \inf_{G^{\#}}\{g \in G : gx_0 \geq x\}, x_0 \in X\right\}$$

Firstly we prove that the set $M(X, G^{\#})_{\sup} \cup M(X, G^{\#})_{\inf}$ is a totally ordered set. For this, it was crucial to analyze the different cases depending on whether G is a quasidiscrete or quasidense group and to considerate the orbits of the G-module X.

Later on, in Section III.2, we describe all G-module maps $X \to G^{\#}$ when X is the G-module X_2 with two orbits (see item (f) in Example 6). This description allows to show that $M(X_2, G^{\#})$ is a totally ordered set.

Finally, using G-module maps in $M(G^{\#}, G^{\#})$, $M(X, G^{\#})_{sup}$ and $M(X, G^{\#})_{inf}$, we prove that $M(X, G^{\#})$ is a totally ordered set and a G-module, for any G-module X.

III.1. The sets $M(X, G^{\#})_{\sup}$ and $M(X, G^{\#})_{\inf}$

Lemma III.1.1.
$M(X, G^{\#})_{sup}$ and $M(X, G^{\#})_{inf}$ are totally ordered sets.

Proof.
If $x_0, x_1 \in X$ with $x_0 < x_1$ and

$$f_{x_0}(x) = \sup_{G^{\#}}\{g \in G : gx_0 \leq x\}$$

$$f_{x_1}(x) = \sup_{G^{\#}}\{g \in G : gx_1 \leq x\}$$

then $f_{x_1} \leq f_{x_0}$. Indeed, we consider the subsets of G

$$A_{x_0}(x) = \{g \in G : gx_0 \leq x\}$$

$$A_{x_1}(x) = \{g \in G : gx_1 \leq x\}$$

If $g \in A_{x_1}(x)$ then $gx_1 \leq x$. Now, since $x_0 < x_1$, and X is a G-module we have that $gx_0 \leq gx_1$. Thus $gx_0 \leq gx_1 \leq x$, $g \in A_{x_0}$ and therefore $A_{x_1} \subseteq A_{x_0}$. With this inclusion, is easy to see that for all $x \in X$, we have that

$$f_{x_1}(x) = \sup_{G^{\#}} A_{x_1}(x) \leq \sup_{G^{\#}} A_{x_0}(x) = f_{x_0}(x)$$

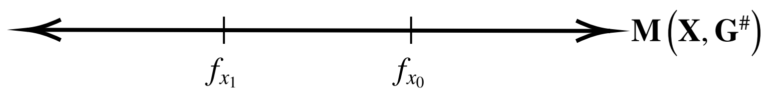


Figure III.1. For $x_0, x_1 \in X$ with $x_0 < x_1$, then $f_{x_1} \leq f_{x_0}$

In the same way, we will prove that $h_{x_1} \leq h_{x_0}$ if $x_0, x_1 \in X$ with $x_0 < x_1$ and

$$h_{x_0}(x) = \inf_{G^{\#}}\{g \in G : gx_0 \geq x\}$$

$$h_{x_1}(x) = \inf_{G^{\#}}\{g \in G : gx_1 \geq x\}$$

In this case, we consider the subsets of G

$$B_{x_0}(x) = \{g \in G : gx_0 \geq x\}$$

$$B_{x_1}(x) = \{g \in G : gx_1 \geq x\}$$

Note that if $g \in B_{x_0}(x)$ then $gx_0 \geq x$. Since $x_0 < x_1$ and X is a G-module then $gx_0 < gx_1$. Thus, $gx_1 > gx_0 \geq x$, $g \in B_{x_1}$ and therefore $B_{x_0} \subseteq B_{x_1}$. Consequently, for all $x \in X$,

$$h_{x_1}(x) = \inf_{G^{\#}} B_{x_1}(x) \leq \inf_{G^{\#}} B_{x_0}(x) = h_{x_0}(x)$$

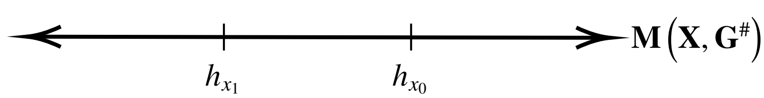


Figure III.2. For $x_0, x_1 \in X$ with $x_0 < x_1$, then $h_{x_1} \leq h_{x_0}$

We conclude that if $x_0 < x_1$, then $f_{x_1}(x) \leq f_{x_0}(x)$ and $h_{x_1}(x) \leq h_{x_0}(x)$, for all $x \in X$. □

Now, our aim is to know if the union $M(X,G^{\#})_{\sup} \cup M(X,G^{\#})_{\inf}$ is a totally ordered set. First for $x_0 \in X$, we compare two G-module maps $f_{x_0} \in M(X,G^{\#})_{\sup}$ and $h_{x_0} \in M(X,G^{\#})_{\inf}$. Note that both maps are generated with the same point $x_0 \in X$,

$$f_{x_0}(x) = \sup_{G^{\#}}\{g \in G : gx_0 \leq x\} \qquad h_{x_0}(x) = \inf_{G^{\#}}\{g \in G : gx_0 \geq x\}$$

First, we calculate $f_{x_0}(x_0)$ and $h_{x_0}(x_0)$.

$$f_{x_0}(x_0) = \sup_{G^{\#}}\{g \in G : gx_0 \leq x_0\} = \sup_{G^{\#}}\ Stab(x_0)$$

$$h_{x_0}(x_0) = \inf_{G^{\#}}\{g \in G : gx_0 \geq x_0\} = \inf_{G^{\#}}\ Stab(x_0)$$

Now, for any $x \in X$, we have the following two cases:

(i) If $x \in Gx_0$ (x is in the orbit of x_0), then there is $g \in G$ such that $x = gx_0$ and as a result

$$f_{x_0}(x) = f_{x_0}(gx_0) = gf_{x_0}(x_0) = g\sup_{G^{\#}} Stab(x_0) = g\sup_{G^{\#}} Stab(x)$$

$$h_{x_0}(x) = h_{x_0}(gx_0) = gh_{x_0}(x_0) = g\inf_{G^{\#}} Stab(x_0) = g\inf_{G^{\#}} Stab(x)$$

Since $\inf_{G^{\#}} Stab(x) \leq \sup_{G^{\#}} Stab(x)$, we have that $g\inf_{G^{\#}} Stab(x) \leq g\sup_{G^{\#}} Stab(x)$ and

$$h_{x_0}(x) \leq f_{x_0}(x) \qquad \forall x \in Gx_0$$

(ii) if $x \notin Gx_0$, then for all $g \in G$ we have that $g \in A_{x_0}(x) = \{g \in G : gx_0 < x\}$ or $g \in B_{x_0}(x) = \{g \in G : gx_0 > x\}$, and

$$A_{x_0}(x) \cap B_{x_0}(x) = \emptyset.$$

Now, if $g \in A_{x_0}(x)$ and $w \in G$ with $w < g$ then $wx_0 \leq gx_0 < x$ and consequently $w \in A_{x_0}(x)$. In the same way, if $g \in B_{x_0}(x)$, and $u \in G$ with $g < u$ then $x < gx_0 \leq ux_0$ and therefore $u \in B_{x_0}(x)$.

This fact implies that $\alpha = \sup_{G^{\#}} A_{x_0}(x) \leq \inf_{G^{\#}} B_{x_0}(x) = \beta$.

If we suppose that $\alpha < \beta$, then there exists $u \in G$ such that

$$\alpha \leq u < \beta \quad \text{or} \quad \alpha < u \leq \beta$$

(i) In the first case, $ux_0 < x$ because $u < \beta$, then $\alpha = u \in G$. Also $\beta \in G$, if we suppose that $\beta \notin G$ then for each $g \in G$ such that $g < \beta$ we have that $gx_0 < x$ and therefore $g \leq \alpha = u$. Thus,

$$\begin{aligned} \beta &= \inf_{G^{\#}}\{g \in G : g \geq \beta\} \\ &= \inf_{G^{\#}}\{g \in G : g > \beta\} \\ &= \sup_{G^{\#}}\{g \in G : g < \beta\} \end{aligned}$$

$$\begin{aligned} &= \sup_{G^{\#}}\{g \in G : g \leq u\} \\ &= u \in G \end{aligned}$$

(ii) Similarly, in the second case, we have that $\alpha, \beta \in G$.

Thus, if $\alpha < \beta$, then $\alpha, \beta \in G$ and we conclude that G must be quasidiscrete with $\beta = g_0\alpha$ where $g_0 = \min\{g \in G : g > 1\}$. This implies that, if G is quasidense then $\alpha = \beta$.

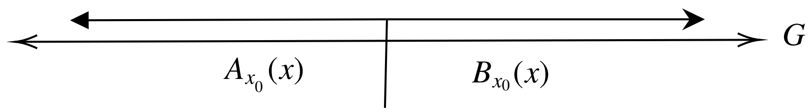


Figure III.3. In the case $x \notin Gx_0$, the subsets $A_{x_0}(x)$, $B_{x_0}(x) \subset G$

Remark 1. Note that, when G is quasidiscrete, if $x \notin Gx_0$ and $\alpha < \beta$ then necessarily $Stab(x_0) = \{1\}$. Indeed, if we suppose that $Stab(x_0) \neq \{1\}$ then $g_0 = \min\{g \in G : g > 1\} \in Stab(x_0)$, therefore

$$x < \beta x_0 = (\alpha g_0)x_0 = \alpha(g_0 x_0) = \alpha x_0 < x,$$

a contradiction.

When we analyze $M(X, G^{\#})_{\sup} \cup M(X, G^{\#})_{\inf}$, it is crucial for the ordering, to know whether G is quasidiscrete or quasidense. The following lemma summarizes this analysis.

Lemma III.1.2.
Let $x_0 \in X$, $f_{x_0} \in M(X, G^{\#})_{sup}$ and $h_{x_0} \in M(X, G^{\#})_{inf}$ G-module maps. In the case G is quasidense, for all $x \in X$ we have that

(i) *If $x \in Gx_0$ and $Stab(x_0) = \{1\}$ then $h_{x_0}(x) = f_{x_0}(x)$.*
(ii) *If $x \in Gx_0$ and $Stab(x_0) \neq \{1\}$ then $h_{x_0}(x) < f_{x_0}(x)$.*
(iii) *If $x \notin Gx_0$ then $h_{x_0}(x) = f_{x_0}(x)$.*

Therefore $h_{x_0} \leq f_{x_0}$.

Remark 2. Note that, If G is quasidense and $Stab(x_0) = \{1\}$ then we do not need to know whether g is in the orbit of x_0 or not, because in both cases $f_{x_0} = h_{x_0}$.

Lemma III.1.3.
Let $x_0 \in X$, $f_{x_0} \in M(X, G^{\#})_{sup}$ and $h_{x_0} \in M(X, G^{\#})_{inf}$ G-module maps. In the case G is quasidiscrete, for all $x \in X$ we have that

(1) If $Stab(x_0) = \{1\}$,
 (i) *If $x \in Gx_0$ then $f_{x_0}(x) = h_{x_0}(x)$.*

(ii) *If* $x \notin Gx_0$ *then* $f_{x_0}(x) \le h_{x_0}(x)$.
Therefore $f_{x_0} \le h_{x_0}$.

(2) If $Stab(x_0) \ne \{1\}$,

(iii) *If* $x \in Gx_0$ *then* $h_{x_0}(x) < f_{x_0}(x)$.
(iv) *If* $x \notin Gx_0$ *then* $h_{x_0}(x) = f_{x_0}(x)$.
Therefore $h_{x_0} \le f_{x_0}$.

Up to now, we have that if G is quasidense and $x_0, x_1 \in X$ with $x_0 < x_1$ then

$$\begin{aligned} f_{x_1} &\le f_{x_0} \\ h_{x_1} &\le h_{x_0} \\ h_{x_0} &\le f_{x_0} \\ h_{x_1} &\le f_{x_1} \\ h_{x_1} &\le f_{x_0} \end{aligned}$$

By the last inequalities, we must compare h_{x_0} and f_{x_1} for $x_0 < x_1$. The next Lemma is true for both cases, quasidense and quasidiscrete.

Lemma III.1.4.
Let G be a totally ordered group and X a G-module. Let $x_0, x_1 \in X$, with $x_0 < x_1$, $f_{x_1} \in M(X, G^{\#})_{sup}$ and $h_{x_0} \in M(X, G^{\#})_{inf}$. Then for all $x \in X$ we have that $f_{x_1}(x) \le h_{x_0}(x)$.

Proof.
In this case we have $A_{x_1}(x) \cap B_{x_0}(x) = \emptyset$. Indeed,

$$\begin{aligned} g \in B_{x_0}(x) &\Rightarrow x \le gx_0 < gx_1 \Rightarrow g \notin A_{x_1}(x) \\ g \in A_{x_1}(x) &\Rightarrow gx_0 < gx_1 \le x \Rightarrow g \notin B_{x_0}(x) \end{aligned}$$

Also, if

$$\begin{aligned} g \in B_{x_0}(x), g < u \in G &\Rightarrow ux_0 \ge gx_0 \ge x \Rightarrow u \in B_{x_0}(x) \\ g \in A_{x_1}(x), u < g \in G &\Rightarrow ux_1 \le gx_1 \le x \Rightarrow u \in A_{x_1}(x) \end{aligned}$$

and it is true that

$$f_{x_1}(x) = \sup_{G^{\#}} A_{x_1}(x) \le \inf_{G^{\#}} B_{x_0}(x) = h_{x_0}(x)$$

□

We summarize the results for the quasidense case.

Theorem III.1.1.
Let G be a quasidense totally ordered group. Then $M(X, G^{\#})_{sup} \cup M(X, G^{\#})_{inf}$ is a totally ordered set.

PROOF.
By Lemma III.1.1, we know that $M(X, G^{\#})_{sup}$ and $M(X, G^{\#})_{inf}$ are totally ordered sets. We just need to show that if $f_y \in M(X, G^{\#})_{sup}$ and $h_z \in M(X, G^{\#})_{inf}$ with $y, z \in X$, then $f_y \leq h_z$ or $f_y \geq h_z$. In fact, if

(i) $y = z$ then, by Lemma III.1.2, we have that $h_z \leq f_y$
(ii) $y > z$ then, by Lemma III.1.4, we have that $f_y \leq h_z$
(iii) $y < z$ then, by Lemma III.1.1, we have that $f_z \leq f_y$ and, by Lemma III.1.2, we know that $h_z \leq f_z$. Therefore $h_z \leq f_y$.

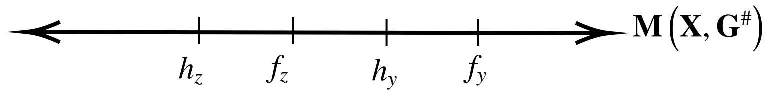


FIGURE III.4. Quasidense case, with $y < z$.

□

Now, let us analyze the quasidiscrete case.

Lemma III.1.5.
Let G be a quasidiscrete totally ordered group with $g_0 := \min\{g \in G : g > 1\}$. Let $x_0, x_1 \in X$, with $x_0 < x_1$ and $f_{x_1} \in M(X, G^{\#})_{sup}$ and $h_{x_0} \in M(X, G^{\#})_{inf}$ G-module maps. Then for all $x \in X$ we have:

(i) *If $Stab(x_0) = \{1\}$ and $Stab(x_1) \neq \{1\}$ then $h_{x_1} \leq f_{x_1} \leq f_{x_0} \leq h_{x_0}$.*
(ii) *If $Stab(x_0) \neq \{1\}$ and $Stab(x_1) = \{1\}$ then $f_{x_1} \leq h_{x_1} \leq h_{x_0} \leq f_{x_0}$.*
(iii) *If $Stab(x_0) \neq \{1\}$ and $Stab(x_1) \neq \{1\}$ then $h_{x_1} \leq f_{x_1} \leq h_{x_0} \leq f_{x_0}$.*
(iv) *If $Stab(x_0) = \{1\}$ and $Stab(x_1) = \{1\}$ then*
  (a) *If $x_0 < x_1 < g_0 x_0$ then $f_{x_1} \leq f_{x_0} \leq h_{x_1} \leq h_{x_0}$.*
  (b) *Otherwise, $f_{x_1} \leq h_{x_1} \leq f_{x_0} \leq h_{x_0}$.*

PROOF.
By Lemma III.1.1, we have that $f_{x_1}(x) \leq f_{x_0}(x)$ and $h_{x_1}(x) \leq h_{x_0}(x)$ for all $x \in X$.

(i) If $Stab(x_0) = \{1\}$ and $Stab(x_1) \neq \{1\}$ then, by Lemma III.1.3, we have that $f_{x_0} \leq h_{x_0}$ and $h_{x_1} \leq f_{x_1}$. Thus $h_{x_1} \leq f_{x_1} \leq f_{x_0} \leq h_{x_0}$.
(ii) If $Stab(x_0) \neq \{1\}$ and $Stab(x_1) = \{1\}$ then, by Lemma III.1.3, we have that $f_{x_1} \leq h_{x_1}$ and $h_{x_0} \leq f_{x_0}$. Thus $f_{x_1} \leq h_{x_1} \leq h_{x_0} \leq f_{x_0}$.
(iii) If $Stab(x_0) \neq \{1\}$ and $Stab(x_1) \neq \{1\}$ then, by Lemma III.1.3 we have that $h_{x_1} \leq f_{x_1}$ and $h_{x_0} \leq f_{x_0}$. Now, we need to compare f_{x_1} with h_{x_0}. Lemma III.1.4, we have that $f_{x_1} \leq h_{x_0}$. Therefore, $h_{x_1} \leq f_{x_1} \leq h_{x_0} \leq f_{x_0}$.
(iv) If $Stab(x_0) = \{1\}$ and $Stab(x_1) = \{1\}$ then, by Lemma III.1.3, we have that $f_{x_1} \leq h_{x_1}$ and $f_{x_0} \leq h_{x_0}$. Now, we need to compare h_{x_1} with f_{x_0}.

(a) If $x_0 < x_1 < g_0x_0$, then x_1 and x_0 are not in the same orbit ($x_1 \notin Gx_0$).
Let $x \in X$ such that $x \in Gx_0$, that is, $x = ux_0$ with $u \in G$. Then $f_{x_0}(x) = f_{x_0}(ux_0) = u$, because $Stab(x_0) = \{1\}$. Also, $x = ux_0 < ux_1$, and thus $h_{x_1}(x) \leq u = f_{x_0}(x)$. Furthermore $h_{x_1}(x) = f_{x_0}(x)$, because if we suppose $h_{x_1}(x) < f_{x_0}(x)$ then $h_{x_1}(x) \leq g_0^{-1}u$, this implies that $x = ux_0 < g_0^{-1}ux_1$ (equality is excluded, because $x_1 \notin Gx_0$), and so $g_0x_0 < x_1$, a contradiction.
Now, if $x \in Gx_1$ then $x = vx_1$ with $v \in G$. Since $Stab(x_1) = \{1\}$ and $vx_0 < vx_1 = u$, we have that $h_{x_1}(x) = v \leq f_{x_0}(x)$. As in the previous case, we have that $h_{x_1}(x) = f_{x_0}(x)$, because $h_{x_1}(x) < f_{x_0}(x)$ then we deduce that $g_0x_0 < x_1$, a contradiction.
Therefore, if $x \in Gx_0 \cup Gx_1$ then $h_{x_1}(x) = f_{x_0}(x)$.
Now, in the case $x \notin (Gx_0 \cup Gx_1)$, there are two possibilities. Firstly, if $A_{x_0}(x) \cap B_{x_1}(x) \neq \emptyset$ then there exists $u \in G$ such that

$$ux_0 < x < ux_1 < ug_0x_0 \quad \text{and}$$

$$g_0^{-1}ux_0 < g_0^{-1}ux_1 < ux_0 < x < ux_1.$$

We have that $ux_0 < x$ and for g_0u, the successor of u in G, $ug_0x_0 > x$, so $f_{x_0}(x) = u$. In the same way $ux_1 > x$ and for $g_0^{-1}u$, the predecessor of u in G, $ug_0^{-1}x_1 < x$, so $h_{x_0}(x) = u$. Therefore $f_{x_0}(x) = h_{x_1}(x)$.
Secondly, if $A_{x_0}(x) \cap B_{x_1}(x) = \emptyset$, then for all $u \in A_{x_0}(x)$ and $v \in B_{x_1}(x)$, we have that

$$ux_0 < ux_1 < x < vx_0 < vx_1.$$

Consequently, $u < v$, and thus $f_{x_0} \leq h_{x_1}$.
We have proved the statement (a).

(b) Now, we will analyze the case $x_0 < g_0x_0 < x_1$. We suppose that there exists $x \in X$ such that $f_{x_0}(x) < h_{x_1}(x)$. First, we have that $A_{x_0}(x) \cap B_{x_1}(x) = \emptyset$, otherwise, there exists $w \in G$ such that $wx_0 \leq x \leq wx_1$ and this implies $h_{x_1}(x) \leq f_{x_0}(x)$, a contradiction to our assumptions. Second, we know that there exists $u \in G$ such that

$$f_{x_0}(x) \leq u < h_{x_1}(x) \quad \text{or} \quad f_{x_0}(x) < u \leq h_{x_1}(x).$$

1. $f_{x_0}(x) \leq u < h_{x_1}(x)$ implies $ux_0 < ux_1 < x$, thus $f_{x_0}(x) = u \in G$ and $h_{x_1}(x) = g_0u \in G$. Thus,

$$ux_0 \leq x < g_0ux_0 \quad , \quad ux_1 < x \leq ug_0x_1$$

and

$$ux_0 < ux_1 < x < g_0ux_0 < g_0ux_1.$$

This leads to $x_0 < x_1 < g_0x_0$, a contradiction.

2. the same happens when $f_{x_0}(x) < u \leq h_{x_1}(x)$, $h_{x_1}(x) = u \in G$ and $f_{x_0}(x) = ug_0^{-1} \in G$. Also, this leads to $x_0 < x_1 < g_0x_0$, a contradiction.

Therefore, for all $x \in X$, $h_{x_1}(x) \leq f_{x_0}(x)$ when $x_0 < g_0 x_0 \leq x_1$ and we have prove the statement (b).

□

We summarize the results for the quasidiscrete case.

Theorem III.1.2.
Let G be a quasidiscrete totally ordered group. Then $M(X, G^{\#})_{sup} \cup M(X, G^{\#})_{inf}$ is a totally ordered set.

Proof.
By Lemma III.1.1, we know that $M(X, G^{\#})_{sup}$ and $M(X, G^{\#})_{inf}$ are totally ordered sets. We just need to show that if $f_y \in M(X, G^{\#})_{sup}$ and $h_z \in M(X, G^{\#})_{inf}$ with $y, z \in X$, then $f_y \leq h_z$ or $f_y \geq h_z$. In fact,

(i) If $y = z$ then, by Lemma III.1.2, we have that
  a) If $\mathrm{Stab}(y) = \mathrm{Stab}(z) = \{1\}$ then $f_y \leq h_z$.
  b) If $\mathrm{Stab}(y) = \mathrm{Stab}(z) \neq \{1\}$ then $h_y \leq f_z$.

(ii) If $y < z$ then, by Lemma III.1.5, we have that
  a) If $\mathrm{Stab}(y) = \mathrm{Stab}(z) = \{1\}$ and $y < z < g_0 y$, then $f_y \leq h_z$.
  b) Otherwise, $h_z \leq f_y$.

(iii) Finally, If $y > z$ then, by Lemma III.1.4, we have that $f_y \leq h_z$.

□

III.2. A totally ordered set of G-module maps

In this section we will show an example of a set G-module maps and we study the ordering on this set for the quasidiscrete as well as the quasidense case. Firstly, let us mention a general fact, which we will use later, about the behavior of any pair of G-module maps.

Remark 3.
Let x_0 be an element of the G-module X. Let r, t be G-module maps in $M(X, G^{\#})$. Note that, if $r(x_0) = t(x_0)$ then $r(x) = t(x)$ for all x in the orbit of x_0. In the same way, if $r(x_0) < t(x_0)$ then $r(x) < t(x)$ for all $x \in Gx_0$.

Indeed, for all $g \in G$

$$r(x_0) = t(x_0) \Rightarrow g \cdot r(x_0) = g \cdot t(x_0) \Rightarrow r(gx_0) = t(gx_0)$$

$$r(x_0) < t(x_0) \Rightarrow g \cdot r(x_0) < g \cdot t(x_0) \Rightarrow r(gx_0) < t(gx_0)$$

III.2.1. The G-module X_2. For this example, we consider a totally ordered group $(G, \cdot, \leq)$ and $G^- := \{g^- : g \in G\}$ be a copy of G disjoint from G. We set $X_2 := G \cup G^-$ and consider the following definitions:

(i) For all $s, t \in G$ with $t < s$, the element $s^- \in G^-$ is such that $t < s^- < s$.
(ii) We extending the multiplication of G to X_2 as follows $g \cdot s^- := (gs)^-$ for all $g \in G$ and $s^- \in G^-$

With these definitions, $(X_2, \cdot, \leq)$ is a G-module with two orbits, namely G and G^-.

Notice, for all $w \in X_2$, we can always write $w = g_w \cdot 1 \in G$ or $w = g_w \cdot 1^- \in G^-$, with $g_w \in G$. Obviously, if $w \in G$ then $w = g_w$.

Now, let $h_w \in M(X_2, G^\#)_{\inf}$ and $f_w \in M(X_2, G^\#)_{\sup}$. For all $x \in X_2$ we have that

- If $w = g_w \cdot 1 \in G$ then

$$\begin{aligned}
f_w(x) &= \sup_{G^\#}\{g \in G : gw \leq x\} \\
&= \sup_{G^\#}\{g \in G : gg_w \leq x\} \\
&= \sup_{G^\#}\{(g_w)^{-1}z \in G : z \leq x\} \quad \text{where } z = gg_w \in G \\
&= (g_w)^{-1} \sup_{G^\#}\{z \in G : z \leq x\} \\
&= (g_w)^{-1} f_1(x)
\end{aligned}$$

$$h_w(x) = (g_w)^{-1} h_1(x)$$

- In the same way, if $w = g_w \cdot 1^- \in G^-$ then $f_w(x) = (g_w)^{-1} f_{1^-}(x)$ and $h_w(x) = (g_w)^{-1} h_{1^-}(x)$.

Moreover, in this example, if $x \in X_2$ then $x = g \in G$ or $x = g \cdot 1^- \in G^-$ and therefore for any G-module map $t \in M(X_2, G^\#)$ we have that $t(g) = gt(1)$ or $t(g^-) = gt(1^-)$. Thus, by the previous equalities, we just need to determine the values of $f_w(1), f_w(1^-)$ and $h_w(1), h_w(1^-)$, with $w = 1, 1^-$.

III.2.2. Case I: G quasidense. We will start determining the G-module maps in $M(X_2, G^\#)_{\sup} \cup M(X_2, G^\#)_{\inf}$ when the totally ordered group G is quasidense.

$$\begin{aligned}
f_1(1) &= \sup_{G^\#}\{g \in G : g \cdot 1 \leq 1\} = 1 \\
f_1(1^-) &= \sup_{G^\#}\{g \in G : g \cdot 1 \leq 1^-\} = \sup_{G^\#}\{g \in G : g < 1^-\} = 1 \\
h_1(1) &= \inf_{G^\#}\{g \in G : g \cdot 1 \geq 1\} = 1 \\
h_1(1^-) &= \inf_{G^\#}\{g \in G : g \cdot 1 \geq 1^-\} = \inf_{G^\#}\{g \in G : g \geq 1^-\} = 1
\end{aligned}$$

$$\begin{aligned}
f_{1^-}(1) &= \sup_{G^\#}\{g \in G : g \cdot 1^- \leq 1\} = \sup_{G^\#}\{g \in G : g^- \leq 1\} = 1 \\
f_{1^-}(1^-) &= \sup_{G^\#}\{g \in G : g \cdot 1^- \leq 1^-\} = \sup_{G^\#}\{g \in G : g^- \leq 1^-\} = 1
\end{aligned}$$

$$\begin{aligned} h_{1^-}(1) &= \inf_{G^{\#}}\{g \in G : g \cdot 1^- \geq 1\} = 1 \\ h_{1^-}(1^-) &= \inf_{G^{\#}}\{g \in G : g \cdot 1^- \geq 1^-\} = \inf_{G^{\#}}\{g \in G : g^- > 1^-\} = 1 \end{aligned}$$

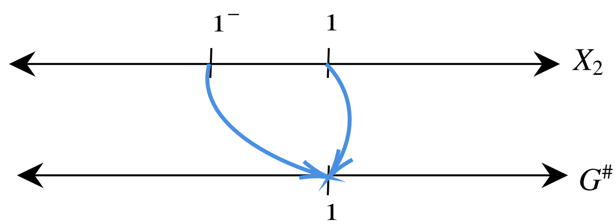


FIGURE III.5. When G is quasidense, $f_{1^-} = f_1 = h_{1^-} = h_1$

Note that, for all $w, x \in X_2$, $h_w(x) = f_w(x) = (g_w)^{-1}g_x$, thus $M(X_2, G^{\#})_{\sup} = M(X_2, G^{\#})_{\inf}$
.

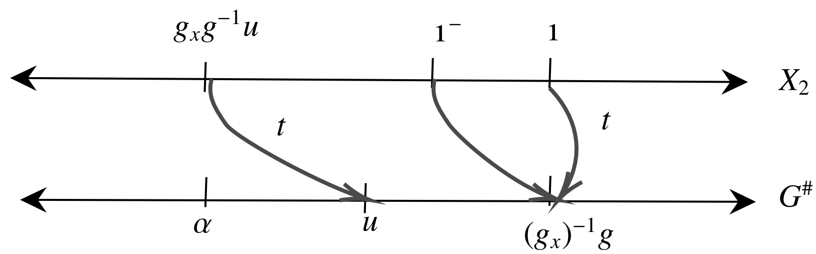
Now, let $t \in M(X_2, G^{\#})$ and we suppose that there exists $x \in X_2$ such that $t(x) = g \in G$.

(i) if $x = g_x \in G$ then $t(g_x) = g$, $t(1) = (g_x)^{-1}g$ and hence

$$t(1) = h_{g_x g^{-1}}(1) = f_{g_x g^{-1}}(1)$$

Therefore $t(u) = h_{g_x g^{-1}}(u) = f_{g_x g^{-1}}(u)$ for all $u \in G$ (see Remark 3).

Now, we show that $t(1^-) = (g_x)^{-1}g$. Indeed, if we suppose $t(1^-) = \alpha < (g_x)^{-1}g$ with $\alpha \in G^{\#}$ then there is $u \in G$ such that $\alpha < u < (g_x)^{-1}g$ (because G is quasidense) and since t is increasing $1^- < ug^{-1}g_x < 1$, which is impossible (note that $t(ug^{-1}g_x) = u$).

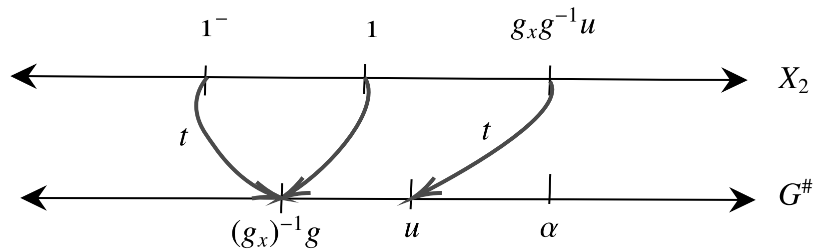


(ii) In the same way, if $x = g_x \cdot 1^- \in G^-$ then $t(g_x^-) = g$, $t(1^-) = (g_x)^{-1}g$ which implies

$$t(1^-) = h_{g_x g^{-1}}(1^-) = f_{g_x g^{-1}}(1^-).$$

Therefore $t(u^-) = h_{g_x g^{-1}}(u^-) = f_{g_x g^{-1}}(u^-)$ for all $u^- \in G^-$ (see Remark 3). Also, $t(1) = (g_x)^{-1}g$. Indeed, if we suppose $t(1) = \alpha > (g_x)^{-1}g$ with $\alpha \in G^{\#}$ then there is $u \in G$ such that $(g_x)^{-1}g < u < \alpha$ (because G is quasidense) and since t is increasing $1^- < ug^{-1}g_x < 1$, which is impossible (note that $t(ug^{-1}g_x) = u$).

In short, if there exists $x \in X_2$ such that $t(x) = g \in G$ then $t = h_w = f_w$ with $w = (g_x)g^{-1}$. Otherwise, $t(x) \in G^{\#} \setminus G$ for all $x \in X_2$.



III.2.3. Case II: G quasidiscrete. We determine the G-module maps in $M(X_2, G^\#)_{\sup} \cup M(X_2, G^\#)_{\inf}$ when G is a quasidiscrete group. Let $g_0 := \min\{g \in G : g > 1\}$.

$$
\begin{aligned}
f_1(1) &= \sup_{G^\#}\{g \in G : g \cdot 1 \leq 1\} = 1 \\
f_1(1^-) &= \sup_{G^\#}\{g \in G : g \cdot 1 \leq 1^-\} = \sup_{G^\#}\{g \in G : g < 1^-\} = g_0^{-1} \\
h_1(1) &= \inf_{G^\#}\{g \in G : g \cdot 1 \geq 1\} = 1 \\
h_1(1^-) &= \inf_{G^\#}\{g \in G : g \cdot 1 \geq 1^-\} = \inf_{G^\#}\{g \in G : g > 1^-\} = 1
\end{aligned}
$$

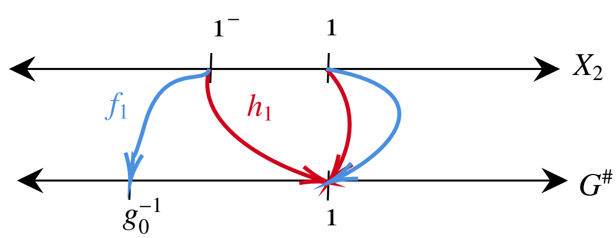


FIGURE III.6. When G is quasidense, $f_1 < h_1$

$$
\begin{aligned}
f_{1^-}(1) &= \sup_{G^\#}\{g \in G : g \cdot 1^- \leq 1\} = \sup_{G^\#}\{g \in G : g^- \leq 1\} = 1 \\
f_{1^-}(1^-) &= \sup_{G^\#}\{g \in G : g \cdot 1^- \leq 1^-\} = \sup_{G^\#}\{g \in G : g^- \leq 1^-\} = 1 \\
h_{1^-}(1) &= \inf_{G^\#}\{g \in G : g \cdot 1^- \geq 1\} = g_0 \\
h_{1^-}(1^-) &= \inf_{G^\#}\{g \in G : g \cdot 1^- \geq 1^-\} = \inf_{G^\#}\{g \in G : g^- \geq 1^-\} = 1
\end{aligned}
$$

Note that, $f_{1^-} = h_1$ and $h_{1^-} = g_0 f_1 = f_{g_0^{-1}}$. Hence, in our analysis we just consider f_w and h_w, with $w \in X_2$. Let $x \in X_2$

- If $x \in Gw$ then $h_w(x) = f_w(x)$
- If $x \notin Gw$ then $f_w(x) < h_w(x)$

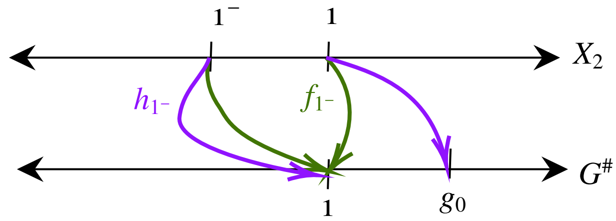


FIGURE III.7. When G is quasidiscrete, $h_{1^-} < f_{1^-}$

Now, let $t \in M(X_2, G^{\#})$ and suppose that there exists $x \in X_2$ such that $t(x) = g \in G$.

(i) if $x = g_x \in G$ then $t(g_x) = g$, $t(1) = (g_x)^{-1}g$ and hence

$$t(1) = h_{g_x g^{-1}}(1) = f_{g_x g^{-1}}(1)$$

Therefore $t(u) = h_{g_x g^{-1}}(u) = f_{g_x g^{-1}}(u)$ for all $u \in G$ (see Observation 3).

Now, since t is increasing, we have two possibilities for $t(1^-)$,

(1) $t(1^-) = (g_x)^{-1}g$ and therefore $t = h_{g_x g^{-1}} \in M(X_2, G^{\#})_{inf}$ or

(2) $t(1^-) = g_0^{-1}(g_x)^{-1}g$ and therefore $t = f_{g_x g^{-1}} \in M(X_2, G^{\#})_{sup}$

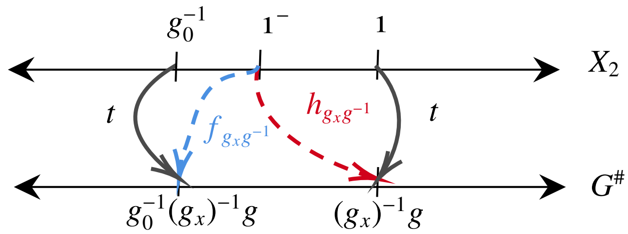


FIGURE III.8. We have two possibilities for $t(1)$. In both cases $t \in M(X_2, G^{\#})_{sup} \cup M(X_2, G^{\#})_{inf}$

(ii) if $x = g_x \cdot 1^- \in G^-$ then $t(g_x \cdot 1^-) = g$, $t(1^-) = (g_x)^{-1}g$ which implies that $t(1^-) = h_{g_x g^{-1}}(1^-)$.

Therefore $t(u^-) = h_{g_x g^{-1}}(u^-)$ for all $u \in G$ (see Remark 3). As in the previous analysis, we have two possibilities for $t(1)$:

(1) $t(1) = (g_x)^{-1}g$ and therefore $t = h_{g_x g^{-1}} \in M(X_2, G^{\#})_{inf}$ or

(2) $t(1^-) = g_0(g_x)^{-1}g$ and therefore $t = f_{g_x g^{-1} g_0^{-1}} \in M(X_2, G^{\#})_{sup}$.

Thus, if there exists $x \in X_2$ such that $t(x) = g \in G$ then $t \in M(X_2, G^{\#})_{inf} \cup M(X_2, G^{\#})_{sup}$. Otherwise, $t(x) \in G^{\#} \setminus G$ for all $x \in X_2$.

III.2.4. $M(X_2, G^{\#})$**: Quasidense and quasidiscrete cases.** Finally, for a quasidense or qusidiscrete totally ordered group G, we analyze the G-module maps $t \in M(X_2, G^{\#})$ such that $t(x) = \alpha \in G^{\#} \setminus G$ for all $x \in X_2$.

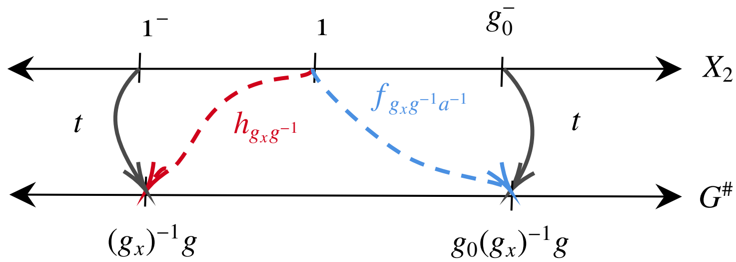


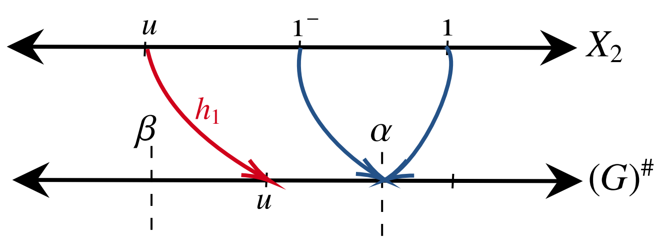
FIGURE III.9. We have two possibilities for $t(1^-)$. In both cases, $t \in M(X_2, G^\#)_{sup} \cup M(X_2, G^\#)_{inf}$

Suppose that t is a G-module map such that $t \notin \left(M(X_2, G^\#)_{sup} \cup M(X_2, G^\#)_{inf}\right)$, with $t(1) = \alpha \in G^\# \setminus G$.

(i) if $\alpha > 1$ then $1 = h_1(1) < t(1) = \alpha$. Also, t is a G-module map, and hence for any $g \in G$, $t(g) = g\alpha$; moreover, $t(g) \leq \alpha$ for all $g \leq 1$. If we suppose that $t(1^-) = \beta < \alpha$ with $\beta \in G^\# \setminus G$, then there is $u \in G$ such that $\beta < u < \alpha$. In this case, $u < 1^-$, $t(u) \leq \beta$, but this is a contradiction because $h_1(g) < t(g)$ for all $g \in G$ (see Remark 3) and therefore

$$u = h_1(u) < t(u) \leq \beta$$

Therefore $t(1^-) = \alpha$ and so $t(g^-) = g\alpha$ for all $g \in G$.



We conclude that $t(x) = g_x\alpha$ for all $x \in X_2$.

(ii) For the case $\alpha < 1$, we have that $t(g) = g\alpha$ for all $g \in G$. If we suppose that $t(1^-) = \beta < \alpha$ with $\beta \in G^\# \setminus G$, then there is $u \in G$ such that $\beta < u < \alpha < 1$. Note that, $u < 1^-$ and hence $t(u) \leq \beta$, which is a contradiction because $u = h_{u^{-1}}(1) < t(1) = \alpha$ but $t(1^-) = \beta < h_{u^{-1}}(1-) = u$ (see Remark 3). Therefore $t(1^-) = \alpha$ and so $t(g^-) = g\alpha$ for all $g \in G$. We conclude that $t(x) = g_x\alpha$ for all $x \in X_2$.

Lemma III.2.1.
$M(X_2, G^\#)$ is a totally ordered set.

PROOF.

First, if G is complete, then $G = G^{\#}$, and

(i) when G is quasidense, $M(X_2, G) = M(X_2, G)_{sup} = M(X_2, G)_{inf}$,
(ii) when G is quasidiscrete, $M(X_2, G) = M(X_2, G)_{sup} \cup M(X_2, G)_{inf}$,
and by Theorems III.1.1, III.1.2 we conclude that $M((X_2, G)$ is a totally ordered set.

Now, suppose that G is not complete, so $G \subsetneq G^{\#}$. Let $t, r \in M(X_2, G^{\#})$ be two G-module maps such that

$$t(1) = t(1^-) = \alpha \quad \text{and} \quad r(1) = r(1^-) = \beta$$

with $\alpha, \beta \in G^{\#} \setminus G$. Then for all $x \in X_2$, $t(x) = g_x\alpha$ and $r(x) = g_x\beta$. Note that if $\alpha < \beta$ then $g_x\alpha < g_x\beta$ for all $g_x \in G$ and therefore $t < r$.

Now, we want to compare the G-module map t with $t(x) = g_x\alpha$ and $\alpha \in G^{\#} \setminus G$ with any G-module map $k \in \left(M(X_2, G^{\#})_{sup} \cup M(X_2, G^{\#})_{inf}\right)$.

(i) When G is quasidense, there exists $w \in X_2$ such that $k(x) = h_w(x) = (g_w)^{-1}g_x$ for all $x \in X_2$. If $k(1) = (g_w)^{-1} < \alpha$ then $k(x) = g_w^{-1}g_x < g_x\alpha = t(x)$ for all $x \in X_2$, and so $k < t$. If $k(1) = (g_w)^{-1} > \alpha$ then $t(x) = g_x\alpha \leq g_w^{-1}g_x = k(x)$ and therefore $t < k$.
(ii) When G is quasidiscrete, $k = h_w$ or $k = f_w$ for some $w \in X_2$.
  (1) In the case $k = h_w$, we have that $k(x) < t(x)$ or $t(x) < k(x)$ (the proof is the same as in the quasidense case).
  (2) When $k = f_w$, we have that $k(1) = f_w(1) = (g_w)^{-1}$ and if $(g_w)^{-1} < \alpha$ then $k(x) \leq (g_w)^{-1}g_x < g_x\alpha = t(x)$ for all $x \in X_2$, and so $k < t$ (remember that $f_w(1^-) = (g_0)^{-1}f_w(1) < f_w(1)$ with $g_0 = \min\{g \in G : g > 1\}$). In the other case, $f_w(1) = (g_w)^{-1} > \alpha$, we have that the predecessor element $(g_0)^{-1}(g_w)^{-1} > \alpha$ and so $f_w(1^-) = (g_0)^{-1}(g_w)^{-1} > \alpha$. Consequently, we have that $t(x) = \alpha g_x < f_w(x) = k(x)$ for all $x \in X_2$ (see Figure III.10).

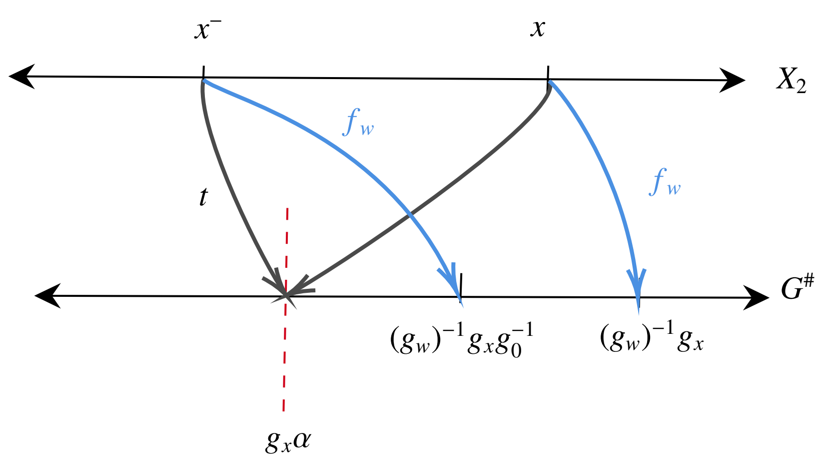


FIGURE III.10. f_w for all $w \in X_2$

□

Knowing the ordering of $M(X_2, G^{\#})_{inf} \cup M(X_2, G^{\#})_{sup}$ was one of the key facts in determining that $M(X_2, G^{\#})$ is a totally ordered set.

In the following section, it is proved that $M(X, G^{\#})$ is a totally ordered set for any G-module X.

III.3. Order in $M(X, G^{\#})$

In [**14**] the set $M(G^{\#})$ of all G-module maps $\varphi : G^{\#} \to G^{\#}$ was described. In the following Lemma, we use the characterization of the G-module maps in the subsets $M^l(G^{\#})$ and $M^r(G^{\#})$ (see definitions and preliminaries in Chapter 1, Section I.5).

Lemma III.3.1.
Let G be a totally ordered group and let X be G-module. For all $\varphi \in M(X, G^{\#})$ and $w \in X$, there are $\phi_{\varphi(w)} \in M^l(G^{\#})$ and $\eta_{\varphi(w)} \in M^r(G^{\#})$ such that for all $x \in X$,

$$\phi_{\varphi(w)}(f_w(x)) \leq \varphi(x) \leq \eta_{\varphi(w)}(h_w(x))$$

where $f_w \in M(X, G^{\#})_{sup}$ and $h_w \in M(X, G^{\#})_{inf}$.

Proof.
Let w be an element in the G-module X. Let $x \in X$. Then for all $g \in G$ such that $g \leq f_w(x)$ we have $g \cdot w \leq x$, hence $g\varphi(w) \leq \varphi(x)$. Therefore $\varphi(x)$ is an upper bound of the set

$$\{g\varphi(w) : g \in G, g \leq f_w(x)\}.$$

Just as before, for $g \in G$ such that $g \geq h_w(x)$ we have that $g\varphi(w) \geq \varphi(x)$ and therefore $\varphi(x)$ is a lower bound of the set

$$\{g\varphi(w) : g \in G, g \geq h_w(x)\}.$$

Thus,

$$\sup_{G^{\#}} \{g\varphi(w) : g \in G, g \leq f_w(x)\} \leq \varphi(x) \leq \inf_{G^{\#}} \{g\varphi(w) : g \in G, g \geq h_w(x)\}.$$

We have that,

$$\sup_{G^{\#}} \{g\varphi(w) : g \in G, g \leq f_w(x)\} = f_w(x) \bullet \varphi(w)$$
$$\inf_{G^{\#}} \{g\varphi(w) : g \in G, g \geq h_w(x)\} = h_w(x) \star \varphi(w)$$

and so, for all $x \in X$,

$$f_w(x) \bullet \varphi(w) \leq \varphi(x) \leq h_w(x) \star \varphi(w).$$

Now, we consider two G module maps in $M(G^{\#})$.

$$\begin{array}{rccc} \eta_{\varphi(w)} : & G^{\#} & \to & G^{\#} \\ & x & \mapsto & x \star \varphi(w) \end{array}$$

$$\begin{array}{rccc} \phi_{\varphi(w)}: & G^{\#} & \to & G^{\#} \\ & x & \mapsto & x \bullet \varphi(w) \end{array}$$

Notice that, $\phi_{\varphi(w)}(f_w(x)) = f_w(x) \bullet \varphi(w)$ and $\eta_{\varphi(w)}(h_w(x)) = h_w(x) \star \varphi(w)$. We conclude that, for all $x \in X$,

$$\phi_{\varphi(w)}(f_w(x)) \leq \varphi(x) \leq \eta_{\varphi(w)}(h_w(x))$$

□

Theorem III.3.1.
$M(X, G^{\#})$ is a totally ordered set.

Proof.
We will prove that if φ_1, φ_2 are G-module maps in $M(X, G^{\#})$ then $\varphi_1 \leq \varphi_2$ or $\varphi_2 \leq \varphi_1$. Let $w \in X$. Then, without loss of generality, we can assume that $\varphi_1(w) < \varphi_2(w)$,

By Lemma III.3.1, for all $x \in X$,

$$\phi_{\varphi_1(w)}(f_w(x)) \leq \varphi_1(x) \leq \eta_{\varphi_1(w)}(h_w(x))$$
$$\phi_{\varphi_2(w)}(f_w(x)) \leq \varphi_2(x) \leq \eta_{\varphi_2(w)}(h_w(x))$$

We will prove that $\eta_{\varphi_1(w)}(h_w(x)) \leq \phi_{\varphi_1(w)}(f_w(x))$. Indeed,

(i) when $h_w(x) \leq f_w(x)$, since η, ϕ are G-module maps in $M(G^{\#})$, we have that

$$\begin{array}{rcll} \varphi_1(x) & \leq & \eta_{\varphi_1(w)}(h_w(x)), & \\ & \leq & \eta_{\varphi_1(w)}(f_w(x)), & \eta \text{ is increasing} \\ & \leq & \eta_{\varphi_2(w)}(f_w(x)), & \varphi_1(w) < \varphi_2(w) \text{ and Proposition I.2.3 (iii)} \\ & \leq & \phi_{\varphi_2(w)}(f_w(x)), & \text{and Proposition I.2.3 (iv)} \\ & \leq & \varphi_2(x) & \end{array}$$

(ii) when $f_w(x) < h_w(x)$, by Lemmas III.1.2 and III.1.3, we have three necessary conditions:

(1.) G must be quasidiscrete,
(2.) $\mathrm{Stab}(w) = \{1\}$ and
(3.) $x \notin Gw$.

In this case, $f_w(x), h_w(x) \in G$ and $h_w(x) = g_0 f_w(x)$ with $g_0 = \min\{g \in G : g > 1\}$ ($h_w(x)$ is the successor of $f_w(x)$ in G).

Using the previous conditions and the definition of $f_w \in M(X, G^{\#})_{sup}$, it we deduce that

$$f_w(x) \cdot w < x < g_0 f_w(x) \cdot w, \qquad \text{for all } x \in X$$

Moreover φ_1, φ_2 are G-module maps, so for all $x \in X$

$$f_w(x) \cdot \varphi_1(w) \leq \varphi_1(x) \leq g_0 f_w(x) \cdot \varphi_1(w),$$
$$f_w(x) \cdot \varphi_2(w) \leq \varphi_2(x) \leq g_0 f_w(x) \cdot \varphi_2(w).$$

If we suppose that $f_w(x)\cdot\varphi_2(w) < g_0 f_w(x)\cdot\varphi_1(w)$ then $\varphi_2(w) < g_0\cdot\varphi_1(w)$ because $f_w(x)\in G$, and $\varphi_1(w) < \varphi_2(w) < g_0\cdot\varphi_1(w)$. Similarly, because $g_0\in G$, we have that $g_0\cdot\varphi_1(w) < g_0\cdot\varphi_2(w)$ and therefore

$$\varphi_1(w) < \varphi_2(w) < g_0\cdot\varphi_1(w) < g_0\cdot\varphi_2(w).$$

The previous inequality implies that $\varphi_1(w),\varphi_2(w)\in G^{\#}\setminus G$, because if $\varphi_1(w)\in G$ then its successor is $g_0\cdot\varphi_1(w) > \varphi_2(x)$, a contradiction. With the same argument we prove that $\varphi_2(x)\notin G$. Thus, there exists $u\in G$ such that $\varphi_1(w) < u < \varphi_2(w)$ and $g_0\cdot\varphi_1(w) < g_0\cdot u < \varphi_2(w)$, but $\varphi_2(w) < g_0\cdot\varphi_1(w)$ and this implies that $u < \varphi_2(w) < g_0\cdot u$, where $g_0\cdot u$ is the successor of $u\in G$, a contradiction. We conclude that $f_w(x)\cdot\varphi_2(w)\geq g_0 f_w(x)\cdot\varphi_1(w)$ and therefore for all $x\in X$

$$\varphi_1(x)\leq g_0 f_w(x)\cdot\varphi_1(w)\leq f_w(x)\cdot\varphi_2(w)\leq\varphi_2(x)$$

□

Theorem III.3.2.
Let G be a totally ordered group and X a G-module. $M(X,G^{\#})$ is a G-module with the action

$$\begin{array}{ccc} G\times M(X,G^{\#}) & \to & M(X,G^{\#}) \\ (g,\varphi) & \mapsto & (g\varphi) \end{array}$$

where $(g\varphi)(x) = g\varphi(x)$ for all $x\in X$ and $\varphi\in M(X,G^{\#})$.

Proof.
We know that $M(X,G^{\#})$ is a totally ordered set and $M(X,G^{\#})\neq\emptyset$ for any G-module X (remember that for any $x_0\in X$, the map $f_{x_0}: X\to G^{\#}$ defined by $f_{x_0}(x) := sup_{G^{\#}}\{g\in G : gx_0\leq x\}$ is in $M(X,G^{\#})$).

Now, by the definition of the action, it is clear that $g_1(g_2\varphi) = (g_1g_2)\varphi$ and $(1\varphi)=\varphi$ for all $g_1,g_2\in G$ and $\varphi\in M(X,G^{\#})$. We prove that $M(X,G^{\#})$ satisfies $(iii),(iv),(v)$ Definition I.4.1.

Let $g,g_1,g_2\in G$, $x\in X$ and $\varphi_1,\varphi_2\in M(X,G^{\#})$.

(iii) If $g_1\leq g_2$ then $g_1x\leq g_2x$ because X is a G-module. For any G-module map $\varphi\in M(X,G^{\#})$, we have that $\varphi(g_1x)\leq\varphi(g_2x)$ because φ is an increasing map, thus $g_1\varphi(x)\leq g_2\varphi(x)$ for all $x\in X$ and therefore $g_1\varphi\leq g_2\varphi$.

(iv) Suppose that $\varphi_1\leq\varphi_2$ (this is possible because $M(X,G^{\#})$ is a totally ordered set), then $g\varphi_1(x)\leq g\varphi_2(x)$ for all $x\in X$, because $\varphi_1(x)$ and $\varphi_2(x)$ belong to the G-module $G^{\#}$. Thus $g\varphi_1\leq g\varphi_2$ for all $g\in G$ and $\varphi\in M(X,G^{\#})$.

(v) Finally, we will prove that for all $\varphi\in M(X,G^{\#})$, the orbit $G\varphi$ is cofinal in $M(X,G^{\#})$, that is to say, for any $\varphi_1\in M(X,G^{\#})$ we can find $g\in G$ such that $\varphi_1\leq g\varphi$.

If $\varphi_1\leq\varphi$ then $g=1\in G$ satisfies the statement. Now, we supppose that there exists $w\in X$ such that $\varphi(w)\leq\varphi_1(w)$ and we consider two G-module maps in $M(G^\#)$

$$\begin{array}{rccc} \eta_{\varphi_1(w)}: & G^\# & \to & G^\# \\ & x & \mapsto & x \star \varphi_1(w) \end{array}$$

$$\begin{array}{rccc} \phi_{\varphi(w)}: & G^\# & \to & G^\# \\ & x & \mapsto & x \bullet \varphi(w) \end{array}$$

and the G-module maps h_w and f_w in $M(X, G^\#)_{inf}$ and $M(X, G^\#)_{sup}$ respectively.

$M(G^\#)$ is a G-module (see Theorem I.5.4) and therefore there exists $g \in G$ such that $\eta_{\varphi_1(w)} \leq g\phi_{\varphi(w)}$. In particular, for $h_w(x) \in G^\#$ we have

$$\eta_{\varphi_1(w)}(h_w(x)) \leq g\phi_{\varphi(w)}(h_w(x))$$

for all $x \in X$. First, If $h_w \leq f_w$ then

$$\eta_{\varphi_1(w)}(h_w(x)) \leq g\phi_{\varphi(w)}(h_w(x)) \leq g\phi_{\varphi(w)}(f_w(x)).$$

By Lemma III.3.1 we have that for all $x \in X$

$$\varphi_1(x) \leq \eta_{\varphi_1(w)}(h_w(x)) \leq g\phi_{\varphi(w)}(f_w(x)) \leq g\varphi(x)$$

and so $\varphi_1 \leq g\varphi$.

Now, if $f_w < h_w$, then G is quasidiscrete and $g_0 f_w = h_w$ (see Lemma III.1.3). Also, since $\eta_{\varphi_1(w)} \leq g\phi_{\varphi(w)}$

$$\eta_{\varphi_1(w)}(f_w(x)) = \eta_{\varphi_1(w)}(g_0^{-1} h_w(x)) \leq g\phi_{\varphi(w)}(f_w(x))$$

$$g_0^{-1}\eta_{\varphi_1(w)}(h_w(x)) \leq g\phi_{\varphi(w)}(f_w(x))$$

$$\varphi_1(x) \leq \eta_{\varphi_1(w)}(h_w(x)) \leq (gg_0\phi_{\varphi(w)}(f_w(x)) \leq (gg_0)\varphi(x).$$

Hence, for all $x \in X$, $\varphi_1(x) \leq (gg_0)\varphi(x)$ and we conclude $\varphi_1 \leq (gg_0)\varphi$.

□

III.4. Future work: G-module maps in $M(X, Y)$

We have shown that for all G-module X the set $M(X, G^\#)$ is a totally ordered group with the order relation $\varphi_1 \leq \varphi_2$ if and only if for all $x \in X$, $\varphi_1(x) \leq \varphi_2(x)$. Now, let Y be another G-module, if we consider the same order relation, is $M(X, Y)$ a totally ordered set? Is $M(X, Y)$ a G-module with the action $G \times M(X, Y) \to M(X, Y)$ given by $(g\varphi)(x) := g\varphi(x) = \varphi(gx)$ for all $x \in X$ and $\varphi \in M(X, Y)$?

In the study of G- module maps, the orbit of an element $x_0 \in X$ played an important role in comparing $r, t \in M(X, G^\#)$ (see Remark 3 in Chapter 3). In order to answer these questions, I am thinking to work with the following two concepts and the results obtained from them.

(i) In [**8**] the concept of *Tight G-module* to describe a G-module X that has a convex base was introduced. B is a convex base of X, when X is generated by a convex subset $B \subset X$ and for all $b_1, b_2 \in B$ the orbits Gb_1 and Gb_2 are disjoint. For example, the G-module X_2 in the Section III.2.1 is a tight with convex base $\{g^-, g\}$ for any $g \in G$.

(ii) The topological type of an element in the G-module X was defined in [**10**] as follows. Let s_0 be an element of X and for each $s \in X$, we consider

$$\tau_l(s) := \sup_{X^\#} \{x \in Gs : x \leq s_0\}$$

$$\tau_u(s) := \sup_{X^\#} \{x \in Gs : x \geq s_0\}$$

Note that these elements of X show to what extent we can approach s_0 with elements of the orbit of s.

The topological type of an element $s \in X$ was defined as the subset of G given by

$$\tau(s) := \{h \in G \ : \tau_l(s) \leq hs_0 \leq \tau_u(s)\}.$$

As a first step, we could study $M(X, Y)$ in the particular case where X or Y are tight G-modules. Let G be a totally ordered group and let X be a tight G-module with a convex base B. Note the followings facts:

- The base B contains one and only one element of each of the orbits Gb with $b \in B$.
- Let $z, w, v \in X$ with $z < w < v$. Note that if $gw \leq v$ for some $g \in G$ then, because B is a convex subset of X, we have that $gw < z$.
- Choose v in the convex base B. For each $w \in B$, we have

$$\tau_l(w) = \sup_{X^\#}\{x \in Gw : x < v\}$$

$$\tau_u(w) = \inf_{X^\#}\{x \in Gw : x > v\}.$$

  If $w = v$, then $\tau_l(w) = \tau_u(w) = w$ and the topological type of w is $\tau(w) = stab(v)$.
  If $w < v$, then $\tau_l(w) = w$ and for all $b \in B$, $\tau_u(w) \geq b$.
  If $h \in stab(v)$, then $w < v = hv \leq \tau_u(w)$. If $h \notin stab(v)$, then $1 < h$ leads to $\tau_u(w) \leq hw < hv$ and $h < 1$ implies that $hw < hv < b$ for all $b \in B$. These facts imply that

$$\tau(w) = \{h \in G : \ w \leq hv \leq \tau_u(w)\} = \mathrm{Stab}(v)$$

  In the same way, we show that $\tau(w) = stab(v)$ when $v < w$.
- Also, in [**8**] the authors proved that if X is a tight G-module, then the stabilizer of an element $x \in X$ is constant convex subgroup $H \subset G$. Therefore, the topological type does not depend on the choice of $v \in B$, is constant on X.

We can to sort the elements of a tight G-module X with base B by considering the sets gB for all $b \in B$. Because for all $g \in G$, gB is also a convex base of X, we have that

$$g_2^{-1}b_i \ < \ g_2^{-1}b_j \ < \ g_1^{-1}b_i \ < \ g_1^{-1}b_j \ < \ b_i \ < \ b_j \ < \ g_1b_i \ < \ g_1b_j \ < \ g_2b_i \ < \ g_2b_j$$

for all $b_i, b_j \in B$ with $b_i < b_j$ and $g_1, g_2 \in G$ two elements that do not belong to $\mathrm{Stab}(b_i) = H$ with $1 < g_1 < g_2$.

Thus, we can analyze the orbits of the elements of the base B of X to compare two G-module maps $r, t : X \to Y$.

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