FACULTAD DE INGENIERÍA Y CIENCIAS

Departamento de Matemática y Estadística

ON THE CONNECTIVITY OF THE BRANCH AND REAL LOCUS OF $\mathcal{M}_{0,[n+1]}$

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Abstract. If $n \geq 3$, then the moduli space $\mathcal{M}_{0,[n+1]}$, of isomorphisms classes of $(n+1)$-marked spheres, is a complex orbifold of dimension $n-2$. Its branch locus $\mathcal{B}_{0,[n+1]}$ consists of the isomorphism classes of those $(n+1)$-marked spheres with non-trivial group of conformal automorphisms. If either (i) $n \geq 4$ is even or (ii) $n \geq 6$ is divisible by 3, then we prove that $\mathcal{B}_{0,[n+1]}$ is connected. Otherwise, we observe that it has exactly two connected components. The orbifold $\mathcal{M}_{0,[n+1]}$ also admits a natural real structure, this being induced by the complex conjugation on the Riemann sphere. The locus $\mathcal{M}_{0,[n+1]}(\mathbb{R})$ of its fixed points, the real points, consists of the isomorphism classes of those marked spheres admitting an anticonformal automorphism. Inside this locus is the real locus $\mathcal{M}^{\mathbb{R}}_{0,[n+1]}$, consisting of those classes of marked spheres admitting an anticonformal involution. We prove that $\mathcal{M}^{\mathbb{R}}_{0,[n+1]}$ is connected for $n \geq 5$ odd, and that it is disconnected for $n = 2r$ with $r \geq 5$ is odd. If n is odd, then in general $\mathcal{M}_{0,[n+1]}(\mathbb{R})$ and $\mathcal{M}^{\mathbb{R}}_{0,[n+1]}$ are different and every connected component of $\mathcal{M}_{0,[n+1]}(\mathbb{R})$ intersects the real locus $\mathcal{M}^{\mathbb{R}}_{0,[n+1]}$, therefore $\mathcal{M}_{0,[n+1]}(\mathbb{R})$ is connected. If n is even we have that $\mathcal{M}_{0,[n+1]}(\mathbb{R}) = \mathcal{M}^{\mathbb{R}}_{0,[n+1]}$.

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CONTENT

Introduction

The moduli space $\mathcal{M}_{g,[r]}$, of isomorphism classes of (r)-marked Riemann surfaces of genus $g \geq 0$, is a complex orbifold of dimension $3g-3+r$, where $r \geq 0$ is a integer (so, for $g=0$, we have $r \geq 3$). Its branch locus $\mathcal{B}_{g,[r]} \subset \mathcal{M}_{g,[r]}$ consists of the isomorphism classes of those admitting non-trivial conformal automorphisms. In [5] it was proved that $\mathcal{B}_{g,[0]} \subset \mathcal{M}_{g,[0]} = \mathcal{M}_g$ is connected only for $g \in \{3,4,13,17,19,59\}$. The complex orbifold $\mathcal{M}_{g,[r]}$ also admits a natural anti-holomorphic automorphism of order two (a real structure) which is induced by the usual complex conjugation. The locus $\mathcal{M}_{g,[r]}(\mathbb{R}) \subset \mathcal{M}_{g,[r]}$ of fixed points (the real points) of such a real structure consists of the isomorphic classes of those admitting anticonformal automorphisms. Let $\mathcal{M}^{\mathbb{R}}_{g,[r]} \subset \mathcal{M}_{g,[r]}(\mathbb{R})$ be the sublocus of those classes having a representative admitting an anticonformal involution (equivalently, the representative being definable over the reals), we call $\mathcal{M}^{\mathbb{R}}_{g,[r]}$ the real locus of $\mathcal{M}_{g,[r]}(\mathbb{R})$. In [11, 16, 48] it has been proved that $\mathcal{M}^{\mathbb{R}}_{g,[0]} \subset \mathcal{M}_g$ is connected. In [20] it was proved that $\mathcal{M}_{g,[0]}(\mathbb{R}) \subset \mathcal{M}_g$ is also connected but that $\mathcal{M}_{g,[0]}(\mathbb{R}) \setminus \mathcal{M}^{\mathbb{R}}_{g,[0]}$ is not in general connected. In this thesis we consider $r = n+1$, $g = 0$, and $n \geq 3$, that is, we work with $(n+1)$-marked spheres and we study the connectivity of the branch locus $\mathcal{B}_{0,[n+1]}$, the real locus $\mathcal{M}^{\mathbb{R}}_{0,[n+1]}$ and the locus the real points $\mathcal{M}_{0,[n+1]}(\mathbb{R})$.

Torelli space $\mathcal{M}_{0,n+1}$ is the moduli space of isomorphisms classes of ordered $(n+1)$-marked spheres. The modular group $\mathrm{Mod}_{0,[n+1]}$ induces an action of the symmetric group $\mathfrak{S}_{n+1}$ as a group $\mathbb{G}_n$ of holomorphic automorphisms of $\mathcal{M}_{0,n+1}$ (called the Torelli group) and $\mathcal{M}_{0,[n+1]} = \mathcal{M}_{0,n+1}/\mathfrak{S}_{n+1}$. If $n=3$, then $\mathcal{M}_{0,4}$ can be identified with the orbifold whose underlying space is $\Omega_3 = \mathbb{C} \setminus \{0,1\}$ and all of its points being singular points of order 4. In this case, $\mathbb{G}_3 \cong \mathfrak{S}_3$ (the action of $\mathfrak{S}_4$ on $\mathcal{M}_{0,4}$ is not faithful as it contains a normal subgroup $K_3 \cong C_2^2$ acting trivially). In particular, $\mathcal{B}_{0,[4]} = \mathcal{M}_{0,[4]}$. The quotient orbifold $\Omega_3/\mathbb{G}_3$ can be identified with the complex plane $\mathbb{C}$ with two cone points, one of order two and the other of order three, (the two cone points corresponds exactly to those 4-marked spheres whose of conformal automorphisms is bigger than C_2^2). Also, $\mathcal{M}^{\mathbb{R}}_{0,[4]} = \mathbb{R}$. If $n \geq 4$, then $\mathcal{M}_{0,n+1}$ can be identified with the domain $\Omega_n \subset \mathbb{C}^{n-2}$ consisting of those tuples $(z_1, \ldots, z_{n-2})$, where $z_j \in \Omega_3$ and $z_i \neq z_j$ for $i \neq j$. In this case, $\mathbb{G}_n \cong \mathfrak{S}_{n+1}$ acts faithfully as the full group of holomorphic automorphisms of $\mathcal{M}_{0,n+1}$ [46, 25] and $\Omega_n/\mathbb{G}_n = \mathcal{M}_{0,[n+1]}$. If $\mathrm{Sing}_{0,[n+1]} \subset \mathcal{M}_{0,[n+1]}$ is the locus of non-manifold points, then: (i) for $n \geq 6$, $\mathrm{Sing}_{0,[n+1]} = \mathcal{B}_{0,[n+1]}$ [43] and (ii) for $n \in \{4,5\}$, the singular locus consists of exactly one point [35]. If, for $T \in \mathbb{G}_n \setminus \{I\}$, we denote by $\mathrm{Fix}(T) \subset \Omega_n$ the locus of its fixed points, then in [47] it was observed that, for $\mathrm{Fix}(T) \neq \emptyset$ (which might not be connected), its projection to $\mathcal{M}_{0,[n+1]}$ is connected.

The structure of this thesis is the following. In the first chapter (Preliminaries), we review our basic study objects, these are: branched covering, complex manifolds, Riemann surfaces, holomorphic and anti-holomorphic automorphisms and complex orbifold. In the second chapter (Moduli and Torelli spaces of marked surfaces) we present the Teichmüller space, the modular group and the pure modular group associated with a Riemann surface, and we obtain the moduli space and the Torelli space as quotient of the Teichmüller space for the respective modular groups, we also present the Ω_n space and its $\mathbb{G}_n$ group of holomorphic automorphisms obtaining to Ω_n and $\Omega_n/\mathbb{G}_n$ as models for the Torelli space and the moduli space respectively, we also discuss the special case for $n = 3$ which is summarized above, and since for the case studies that we interest, that is $n \geq 4$ we have that the modular group is the full group of holomorphic automorphisms of the Teichmüller space, we have that $\mathbb{G}_n$ is the full group of holomorphic automorphisms of the Torelli space, with this an anti-holomorphic automorphism of Ω_n is presented as $T \circ J$ with $T \in \mathbb{G}_n$ and J is the anti-holomorphic automorphism of order two induced by the complex conjugation, finally we define our main objects of study the branch locus $\mathcal{B}_{0,[n+1]}$, the real locus $\mathcal{M}^{\mathbb{R}}_{0,[n+1]}$ and the locus the real points $\mathcal{M}_{0,[n+1]}(\mathbb{R})$. In the third chapter (Results) we present the main results of this investigation, the Theorem 6 refers to the connectivity of the branch locus, obtaining as main result that the branch locus is connected if n is even or if $n \geq 6$ is divisible by 3, and in the others cases has exactly two connected components, the Theorem 7 refers to the connectivity of the real locus, obtaining as main result that the real locus $\mathcal{M}^{\mathbb{R}}_{0,[n+1]}$ is connected for $n \geq 5$ odd and it is not connected for $n = 2r$, $r \geq 5$ odd, also if $p \geq 5$ is a prime, then $\mathcal{M}^{\mathbb{R}}_{0,[2p+1]}$ has exactly $(p-1)/2$ connected components, the connectivity graph of the irreducible components of the real locus is also presented, determining if the real locus is connected or not for each $n \geq 4$. We observed that for the locus of the real points, $\mathcal{M}_{0,[n+1]}(\mathbb{R})$, when n is odd, then in general $\mathcal{M}_{0,[n+1]}(\mathbb{R})$ and $\mathcal{M}^{\mathbb{R}}_{0,[n+1]}$ are different, and every connected component of $\mathcal{M}_{0,[n+1]}(\mathbb{R})$ intersects the real locus $\mathcal{M}^{\mathbb{R}}_{0,[n+1]}$, therefore $\mathcal{M}_{0,[n+1]}(\mathbb{R})$ is connected, if n is even we have that $\mathcal{M}_{0,[n+1]}(\mathbb{R}) = \mathcal{M}^{\mathbb{R}}_{0,[n+1]}$. Finally in the fourth chapter (Applications) we study the Moduli spaces of the generalized Fermat curves of type (k, n) and the hyperelliptic Riemann surfaces of genus g, the same ones that are modeled by the quotient orbifold $\Omega_n/\mathbb{G}_n$, so, we have that the locus in $\mathcal{F}_{k,n}$ (Moduli space of generalized Fermat curves of type (k, n)) , consisting of those admitting more conformal automorphisms than the generalized Fermat group of the (k, n), is connected for $n \geq 4$ even and for $n \geq 6$ divisible by 3, and it has exactly two connected components otherwise. Its real locus is connected for $n \geq 5$ odd, and it is not connected for $n = 2r$, $r \geq 5$ odd, analogously, the locus in $\mathcal{H}_g$ (Moduli space of hyperelliptic Riemann surfaces), consisting of those hyperelliptic Riemann surfaces admitting more conformal automorphisms than the hyperelliptic one, is connected if $2g + 1$ is divisible by 3 and it has exactly two connected components otherwise, and the real locus in $\mathcal{H}_g$ is connected. This results can be applied as well to the more unknown (and difficult to work with) generic p-gonal curves, simple generic p-gonal curves [**17**, **18**, **39**, **21**].

CHAPTER I

Preliminaries

This chapter is devoted to summarize the material that constitutes the background for the rest of the thesis. The contents of Sections are well-known facts about branched coverings, complex manifolds and complex orbifolds.

I.1. Branched Covering

All of our spaces will be in general Hausdorff, arc-connected and locally arc-connected (unless we indicate otherwise).

Definition 1 (Branched Covering)**.** *Let X and Y be topological spaces and $p : X \to Y$ be a continuous and surjective function. It is said that p is a* **branched covering** *if for each $y \in Y$ there is an open $U \subset Y$, $y \in U$, such that:*

(1) $p^{-1}(U)$ is a disjoint union of open sets $V_i \subset X$, $i \in I_y$;
(2) for each $i \in I_y$, there is a finite subgroup $\Gamma_i < Homeo(V_i)$, where $Homeo(V_i)$ is the homeomorphisms group of V_i; and
(3) for each $i \in I_y$, the restriction $p|_{V_i} : V_i \to U$ is surjective and given $x_1, x_2 \in V_i$, we have that, $p(x_1) = p(x_2) \Leftrightarrow \exists \gamma \in \Gamma_i$ such that $x_2 = \gamma(x_1)$;
(4) the locus of points in X with non-trivial Γ_i a nowhere dense.

Remark 1. Let $p : X \to Y$ be a branched covering.

(1) If in the above definition, for given $y \in Y$ and every $i \in I_y$, we have that Γ_i is the trivial group $\{id : x \mapsto x\}$, then p is a covering between topological spaces.
(2) Let $y \in Y$ and U an open neighborhood of y as in the definition 1. Then the following hold:
   (a) In each connected component V_i of $p^{-1}(U)$ we have that $\#(V_i \cap p^{-1}(y))$ consists of a finite collection of points, all of them in the same orbit for Γ_i; denote by q_{V_i} one of such points.
   (b) If the point $q_{V_i} \in p^{-1}(y)$ has Γ_i- non-trivial stabilizer, then this is a **branching point** of p and $p(q_{V_i})$ is a **branching value** of p (also called a **singular point** of Y). The **order** of q_{V_i} is the order of its Γ_i-stabilizer. A singular point associated to a cyclic stabilizer is also called a **conical point**.

(3) If the branched covering p has finite degree, that is, $\forall y \in Y$, $\#p^{-1}(y)$ is finite, then we can define the **branching order** of $y \in Y$, as the least common multiple of all the corresponding orders of the stabilizers of their pre-images.

Definition 2 (Deck group)**.** *Let X and Y topological spaces and $p : X \to Y$ a branched covering. The group* $\mathrm{deck}(p : X \to Y) := \{f : X \longrightarrow X : f \in Homeo(X),\ p = p \circ f\}$, *it's called the* **Deck group** *(or cover group) of p.*

Example 1. A covering between topological spaces is a branched covering whose set of branching values is empty.

Definition 3 (Properly discontinuous action)**.** *Let X be a topological space and $\Gamma < Homeo(X)$ be a subgroup of homeomorphisms of X. We say that Γ act* **properly discontinuously** *on X, if $\forall x \in X$ the following properties hold:*

(1) its stabilizer $\Gamma_x = \{\gamma \in \Gamma : \gamma(x) = x\}$ is finite, and
(2) there is an open $V_x \subset X$ with $x \in V_x$, such that $\gamma(V_x) \cap V_x = \emptyset,\ \forall \gamma \in \Gamma - \Gamma_x$.

Note that we can assume that the open V_x satisfies the following property:

$$\gamma(V_x) = V_x \quad \forall \gamma \in \Gamma_x.$$

Example 2 (Galois branched covering)**.** Let's assume Γ acts properly discontinuously on X. Then in X we define the equivalence relation:

$$x_1 \sim_\Gamma x_2 \Leftrightarrow \exists \gamma \in \Gamma : x_2 = \gamma(x_1).$$

Denote by X/Γ the set of equivalence classes $[x]_\Gamma$, and $p : X \to X/\Gamma : x \mapsto [x]_\Gamma$ to the quotient application. Giving X/Γ of the quotient topology, we have that p is a continuous and open application.

Note that we may have that X/Γ might not be Hausdorff. In order to get the Hausdorff condition, we also need the following property:

(H) for every two points $x_1, x_2 \in X$, which are not Γ-equivalent, there are neighborhoods V_1 of x_1 and V_2 of x_2 such that $\gamma_1(V_1) \cap \gamma_2(V_2) = \emptyset$, for every $\gamma_1, \gamma_2 \in \Gamma$ (see next example below).

Let $[x]_\Gamma \in X/\Gamma$. By hypothesis we have that Γ_x is finite and that there is an open V_x that contains x and that it satisfies the conditions of the definition of properly discontinuous. We take $U = p(V_x) \subset X/\Gamma$ which is an open that contains $[x]_\Gamma$. As $p^{-1}(U) = \cup_{\gamma \in \Gamma/\Gamma_x} \gamma(V_x)$ and the group $\gamma \circ \Gamma_x \circ \gamma^{-1} < Homeo(\gamma(V_x))$ is finite, we note that $p : X \to Y$ meets conditions: (1), (2) and (3) of the definition 1. Therefore, p is a branched covering; called **regular or Galois**, defined by the action of the group $\Gamma < Homeo(X)$.

In this case we have that $[x]_\Gamma$ is a branching value if and only if $\Gamma_x \neq \{id : z \mapsto z\}$, the branching order of $[x]_\Gamma$ is $\#\Gamma_x$.

Example 3. Let $X = \mathbb{R}^2 \setminus \{(0,0)\}$ and $\Gamma = \langle A(x,y) = (x/2, 2y) \rangle \cong \mathbb{Z}$. It can be seen that Γ acts properly discontinuos on X and, moreover, the stabilizer of every point is trivial. But in this case X/Γ is not Hausdorff.

I.2. Complex Manifolds

Definition 4 (Complex Manifold)**.**

(1) An **n-dimensional complex structure** *on a connected, Hausdorff and second countable topological space M is a maximal collection*

$$\mathcal{A} = \{(U_\alpha, \phi_\alpha) : \alpha \in I\},$$

verifying the following properties:

(a) Each U_α is an open set of M.

(b) $M = \bigcup_{\alpha \in I} U_\alpha$.

(c) Each ϕ_α is a homeomorphism between U_α and an open set in $\mathbb{C}^n$. We say that ϕ_α is a **chart local** *of M.*

(d) If $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a biholomorphism, i.e., it is bijective and holomorphic in each of the n variables separately at every point.

(2) An **n-dimensional complex manifold** *is a second countable connected Hausdorff topological space together with an n-dimensional complex structure.*

I.2.1. Function on Complex Manifold.

Definition 5 (Holomorphic and anti-holomorphic functions on Complex Manifold)**.** *Let $f : M_1 \to M_2$ be a function between two complex manifolds. We say that the function f is* ***holomorphic*** *(respectively* ***anti-holomorphic****) if for each $p \in M_1$, there are local charts $\phi_1 : U_1 \to V_1$ for M_1 and $\phi_2 : U_2 \to V_2$ for M_2, such that $p \in U_1$, $f(U_1) \subset U_2$ and $\phi_2 \circ f \circ \phi_1^{-1} : V_1 \subset \mathbb{C} \to \mathbb{C}$ is holomorphic (respectively anti-holomorphic), i.e., holomorphic (respectively anti-holomorphic) in each of the n variables separately at every point.*

Definition 6 (Biholomorphism and anti-biholomorphism on Complex Manifold)**.** *Let $f : M_1 \to M_2$ be a holomorphic (respectively anti-holomorphic) and bijective function between two complex manifolds, we call the function f a* ***biholomorphism*** *(respectively* ***anti-biholomorphism****). If there exists an biholomorphic between M_1 and M_2, we say that M_1 and M_2 are biholomorphics.*

Remark 2. Let us observe, in the above definition, that the inverse of a biholomorphism (respectively, anti-biholomorphism) is also a biholomorphism (respectively, anti-biholomorphism).

Definition 7 (Holomorphic automorphism and anti-holormorphic automorphism of Complex Manifold)**.** *Let M be a complex manifold and $f : M \to M$ a biholomorphism (respectively a anti-biholomorphism), we call to the function f a* ***holomorphic automorphism***

*(respectively **anti-holomorphic automorphism**) on complex manifolds. We denote with $Aut^+(M)$ (respectively $Aut(M)$) to the group (with the composition operation) of holomorphic automorphism of M (respectively to the group of the holomorphic automorphism and anti-holomorphic automorphism).*

I.2.2. Examples of Complex Manifold.

(1) Every connected and non-empty open subset $\Omega \subset \mathbb{C}^n$, for instance, $\mathbb{C}^n$.

(2) Let M be a n-dimensional complex manifold and let G be a group of holomorphic automorphisms of M, acting properly discontinuously. Let us assume that for each point $p \in M$ there must be an open set $U \subset M$, $p \in U$, such that $g(U) \cap U = \emptyset$ for all $g \in G - \{I\}$ (that is, G acts without fixed points). We also assume the property (H) above (in order to get the Hausdorff condition on the quotient space M/G). In this case, M/G turns out to be a n-dimensional complex manifold where the charts can be chosen as the composition of local inverses of the natural projection $\pi : M \to M/G$ with local charts of M.

I.2.3. Riemann Surfaces. A Riemann surface is a 1-dimensional complex manifold (details regarding Riemann surfaces consult [**40**, **26**]).

Remark 3. A Riemann surface is an orientable surface. So if we have a compact Riemann surface, according to the classification of compact orientable surfaces, each of these is a connected sum of g tori for some unique interger $g \geq 0$. This integer g is called the **genus** of Riemann surface.

I.2.3.1. *Examples of Riemann Surfaces.* Below we present some examples of Riemann surfaces, which will be used in this work later on.

Example 4. The connected open sets of the complex plane are Riemann surfaces, provided with the identity as the only local chart, in particular the complex plane $\mathbb{C}$, the unitary disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and the upper half-plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} : \ y > 0\}$ are Riemann surfaces. Moreover every connected open subset of a Riemann surface is a Riemann surface.

Example 5. Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the one-point compactification of the complex plane. The local charts

$$\left\{(\phi_1 : \mathbb{C} \to \mathbb{C} : z \mapsto z)\,,\ \left(\phi_2 : \widehat{\mathbb{C}} - \{0\} \to \mathbb{C} : z \mapsto \frac{1}{z}\right)\right\},$$

has as transition function $\phi_2 \circ \phi_1^{-1} : \mathbb{C} - \{0\} \mapsto \mathbb{C} - \{0\}$ defined by $\phi_2 \circ \phi_1^{-1}(z) = \frac{1}{z}$, which is holomorphic. Thus, $\widehat{\mathbb{C}}$ is a Riemann surface, called the Riemann sphere.

Example 6. Let w_1 and w_2 be two linearly independent $\mathbb{C}$ vectors with respect to $\mathbb{R}$. Consider the following set called lattice,

$$L = \{nw_1 + mw_2 \mid n, m \in \mathbb{Z}\},$$

this set is a discrete subgroup of $\mathbb{C}$ and is isomorphic to $\mathbb{Z}\times\mathbb{Z}$. Consider in $\mathbb{C}$ the equivalence relation, let $z_1, z_2 \in \mathbb{C}$, $z_1 \sim z_2 \Leftrightarrow z_1 - z_2 \in L$ and let $\pi : \mathbb{C} \to T := \mathbb{C}/L$ the canonical projection that gives T the topology quotient, so π is continuous and surjective. Also π is open since for all open set V of $\mathbb{C}$, its image $\pi(V)$ is open if and only if $\pi^{-1}(\pi(V))$ is open from $\mathbb{C}$. Since

$$\pi^{-1}(\pi(V)) = \cup_{w\in L}(V + w)$$

is an union of translators of V, which are all open, it is open. For any $z \in \mathbb{C}$, define the closed parallelogram

$$P_z = \{z + \lambda_1 w_1 + \lambda_2 w_2/\lambda_i \in [0, 1], i = 1, 2\},$$

any point of $\mathbb{C}$ is congruent modulo L to a point of P_z, since P_z is compact and π maps P_z onto X, X is compact.

Since L is discrete then there is $\varepsilon > 0$ such that $|w| > \varepsilon$ for all $w \in L \setminus \{0\}$ then

$$\forall z_0 \in \mathbb{C}, \quad \pi|_{B(z_0,\varepsilon)} : B(z_0, \varepsilon) \to \pi(B(z_0, \varepsilon))$$

it is a local homeomorphism (with respect to T). Consider $V_{z_0} = B(z_0, \varepsilon)$, $U_{z_0} = \pi(B(z_0, \varepsilon))$ and $\phi_{z_0} : U_{z_0} \to V_{z_0}$, that is, the local inverse with respect to $\mathbb{C}$ of π restricted to $B(z_0, \varepsilon)$. We see that (ϕ_{z_0}) is a local chart in $T = \mathbb{C}/L$. If $U = \pi(B(z_1, \varepsilon_1)) \cap \pi(B(z_2, \varepsilon_2))$ is not empty the transition function $\phi_2 \circ \phi_1^{-1}(z) = z + w$ for some fixed $w \in L$ is holomorphic. Thus, T is a compact Riemann surface of genus 1 called complex torus.

I.2.3.2. *Function on Riemann Surfaces.* Since a Riemann surface is a 1-dimensional complex manifold, we can define functions on Riemann surfaces in a similar way, these are: holomorphic and anti-holomorphic functions, biholomorphisms and anti-biholomorphisms and holomorphic automorphisms and anti-holomorphic automorphisms.

Example 7. The upper half-plane $\mathbb{H}$ is biholomorphic to the unitary disc $\mathbb{D}$, the biholomorphism is given by:

$$T : \begin{cases} \mathbb{H} & \to \mathbb{D}, \\ z & \mapsto \quad (z - i)/(z + i). \end{cases}$$

Example 8. The holomorphic automorphisms of the Riemann sphere $\widehat{\mathbb{C}}$ are the Möbius transformations, these are rational functions of the form:

$$T(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. The anti-holomorphic automorphisms are obtained as composition of a Möbius transformations with the complex conjugation ($z \mapsto \overline{z}$). We have that $Aut^{+}(\widehat{\mathbb{C}}) \cong PSL_2(\mathbb{C})$. The group of holomorphic automorphisms of upper half-plane $\mathbb{H}$ is the group:

$$Aut^{+}(H) = PSL_2(\mathbb{R}).$$

Example 9. The group of holomorphic automorphisms of the complex plane $\mathbb{C}$ is the group:

$$Aut^{+}(\mathbb{C}) = \{z \mapsto az + b \mid a, b \in \mathbb{C}, a \neq 0\}.$$

The anti-holomorphic automorphisms are obtained as composition of a element of $Aut^{+}(\mathbb{C})$ with the complex conjugation.

Example 10 (Holomorphic branched coverings between Riemann surfaces)**.** If in the definition 1 we assume that: (i) X, Y are Riemann surfaces, and (ii) $p : X \to Y$ is a holomorphic function, then we can see that for each $V_i \subseteq X$ and $U \subseteq Y$ we can choose biholomorphisms to the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and each $\Gamma_i < Aut^{+}(V_i)$ is a finite cyclic subgroup of holomorphic automorphisms of V_i. In this case we can find $\Phi : U \to \mathbb{D}$ and $\Psi_i : V_i \to \mathbb{D}$ such that $\Pi_{n_i} \circ \Psi_i = \Phi \circ p$, where $\Pi_{n_i} : \mathbb{D} \to \mathbb{D} : z \mapsto z^{n_i}$, see figure: I.1. Thus, p is a **holomorphic branched covering**. In addition, the elements of the deck group of this covering are biholomorphisms.

$$\begin{tikzcd}
V_i \arrow[r, "\Psi_i"] \arrow[d, "p"] & \mathbb{D} \arrow[d] & z \arrow[d, mapsto] \\
U \arrow[r, "\Phi"'] & \mathbb{D} & z^{n_i}
\end{tikzcd}$$

Figure I.1. Holomorphic branched covering

Example 11. Let $\widetilde{S}, S$ be Riemann surfaces and $p : \widetilde{S} \to S$ a holomorphic surjective function of finite degree (that is, the pre-image of each point is finite), then p is a holomorphic branched covering. For example, this happens if $\widetilde{S}$ is compact.

I.2.3.3. *Algebraic Curves.* In this section we will describe examples of Riemann surfaces as zeros of polynomials in affine and proyective spaces.

Example 12 (Smooth affine plane curves)**.** An **affine plane curve** is the locus of zeroes in $\mathbb{C}^2$ of a polynomial $f \in \mathbb{C}[z,w]$. A polynomial $f(z,w)$ is **nonsingular** at a root p if either partial derivative $\partial f/\partial z$ or $\partial f/\partial w$ is not zero at p. The affine plane curve $X = \{(z,w) \in \mathbb{C}^2 \mid f(z,w) = 0\}$ is **nonsingular** or **smooth**, if it is nonsingular at each of its points.

We can obtain complex charts on a smooth affine plane curve using the Implicit function theorem to conclude that the curve is locally a graph. Let $p = (z_0, w_0) \in X$, if $\partial/\partial w(p) \neq 0$, by the Implicit function theorem there is a function $g_p(z)$ such that in a neighborhood U of p, X is the graph $w = g_p(z)$, thus the projection $\pi_z : U \to \pi_z(U) \subset \mathbb{C} : (z,w) \mapsto z$ is a complex chart on X. If instead $\partial f/\partial z(p) \neq 0$, then we make the identical construction using the other projection $\pi_w : (z,w) \mapsto w$ near p, therefore X has a complex structure.

If $f(z,w)$ is an irreducible polynomial (that is, that f cannot be factored nontrivially as $f = g(z,w)h(z,w)$, where both g and h are nonconstant polynomials), then its locus of roots X is connected. Hence if f is nonsingular and irrreducible, X is a Riemann surface.

Definition 8 (The projective n-space)**.** *The* **projective n-space** *denoted by $\mathbb{P}^n$ is the space of the one-dimensional subspaces of $\mathbb{C}^{n+1}$,*

$$\mathbb{P}^n = \{W \subseteq \mathbb{C}^{n+1} | \dim(W) = 1\}.$$

The span of the vector $(x_1, \ldots, x_{n+1}) \in \mathbb{C}^{n+1} \setminus \{0\}$, i.e, $W =< (x_1, \ldots, x_{n+1}) >$ is denoted by $[x_1 : \ldots : x_{n+1}]$; these are the **homogeneous coordinates** of the corresponding point W of $\mathbb{P}^n$. We have that $\mathbb{P}^n$ can also be defined as the set of orbits of $\mathbb{C}^{n+1} \setminus \{0\}$ by the multiplicative action of $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$,

$$\mathbb{P}^n := \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^*,$$

so the projection $\pi_n : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ induces a Haussdorf topology on the projective n-space.

An n-dimensional complex structure for $\mathbb{P}^n$ is given by the open sets:

$$U_i = \{[x_1 : \ldots : x_{n+1}] | \, x_i \neq 0\},$$

for $i = 1, \ldots, n+1$, together with the homeomorphisms:

$$\phi_i : U_i \to \mathbb{C}^n : [x_1 : \ldots : x_{n+1}] \mapsto (x_1/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_{n+1}/x_i),$$

thus, the projective n-space is an n-dimensional complex manifold.

Example 13 (The projective line)**.** If $n = 1$ we have that $\mathbb{P}^1$ denotes **the projective line**, which is a Riemann surface compact and simply connected. It is biholomorphic to the Riemann sphere $\widehat{\mathbb{C}}$; for example with the following correspondence:

$$\begin{array}{ccc} \mathbb{P}^1 & \to & \widehat{\mathbb{C}} \\ [x_1 : x_2] & \mapsto & \begin{cases} x_1/x_2, & x_2 \neq 0 \\ \infty, & x_2 = 0 \end{cases} \end{array}.$$

A polynomial F is **homogeneous** if every term has the same degree in the variables. For example, $F(x,y,z) = y^2z + 5xyz + 9x^3$ is homogeneous of degree 3.

Let $F(x_1,\dots,x_{n+1}) \in \mathbb{C}[x_1,\dots,x_{n+1}]$ be a homogeneous polynomial of degree d in the indeterminate $x_1,\dots,x_{n+1}$ and with complex coefficients. It is not possible to evaluate F in points of $\mathbb{P}^n$ given the non-uniqueness of the homogeneous coordinates, that is, $F(\alpha x_1,\dots,\alpha x_{n+1}) = \alpha^d F(x_1,\dots,x_{n+1})$, $\forall \alpha \in \mathbb{C} \setminus \{0\}$, but if it makes sense write: $\{[x_1 : \dots : x_{n+1}] \in \mathbb{P}^n \mid F(x_1,\dots,x_{n+1}) = 0\}$.

Definition 9 (Projective algebraic curve)**.** *Let $C \subset \mathbb{P}^n$ the common locus of zeros of a collection $\{F_i\}_{i=1,\dots,n-1}$ of homogeneous polynomials, i.e.,*

$$C = \{[x_1 : \dots : x_{n+1}] \in \mathbb{P}^n \mid F_1(x_1,\dots,x_{n+1}) = \dots = F_{n-1}(x_1,\dots,x_{n+1}) = 0\}.$$

We say $C \subset \mathbb{P}^n$ is a smooth complete intersection curve in $\mathbb{P}^n$ or projective algebraic curve, if the Jacobian matrix

$$J_{F_1,\dots,F_{n-1}}(x_1,\dots,x_{n+1}) = \left(\frac{\partial F_i}{\partial x_j}\right) \in M((n-1)\times(n+1), \mathbb{C}[x_1,\dots,x_{n+1}])$$

has maximal rank $n-1$ at every point of C. If $x \in C$ does not have a maximal range, then we will say that x is a **singular point of** *C.*

Theorem 1. *([40]) Every smooth projective algebraic curve is a compact Riemann surface.*

Example 14 (Smooth projective plane curve)**.** Let $F(x,y,z) \in \mathbb{C}[x,y,z]$ be a homogeneous polynomial, and the smooth projective plane curve:

$$C = \{[x : y : z] \in \mathbb{P}^2 \mid F(x,y,z) = 0\}.$$

Let $[x_0, y_0, z_0] \in C$ and suppose that $\dfrac{\partial F}{\partial x}(x_0,y_0,z_0) \neq 0$. Since C is smooth, by the Implicit function theorem we can find a holomorphic function $h(y,z)$ defined in a neighborhood of (y_0,z_0) such that $h(y_0,z_0) = x_0$ and $F(h(y,z),y,z) = 0$, this is easily done for the other two partial derivatives as well. So, we define the complex local charts analogously to the example 12:

$$\begin{array}{rcll} \phi_1 : U_1 = \{[x:y:z] \mid x \neq 0\} & \to & \phi_1(U_1) \subset \mathbb{C} : & [x:y:z] \mapsto (y/x) \\ \phi_2 : U_2 = \{[x:y:z] \mid y \neq 0\} & \to & \phi_2(U_2) \subset \mathbb{C} : & [x:y:z] \mapsto (z/y) \\ \phi_3 : U_3 = \{[x:y:z] \mid z \neq 0\} & \to & \phi_3(U_3) \subset \mathbb{C} : & [x:y:z] \mapsto (y/z) \end{array}.$$

Hyperelliptic curves and generalized Fermat curves are examples of smooth projective algebraic curves, whose definition and properties will be discussed in chapter IV.

As a consequence of the Theorem 1 and the Riemann-Roch theorem (see [40]) we have a correspondence between compact Riemann surfaces and smooth projective curves.

I.3. Complex Orbifold

Definition 10. *An* n-**dimensional complex orbifold** *consists of a second countable connected Haussdorff topological space X (called the* **underlying topological space of the orbifold***) and of a collection*

$$(U_\alpha, V_\alpha, G_\alpha, f_\alpha : U_\alpha \to V_\alpha/G_\alpha); \alpha \in I,$$

satisfying the following properties:

(1) *the collection $\{U_\alpha; \alpha \in I\}$ is an open covering of X;*
(2) *G_α is a finite group of homeomorphisms from the open $V_\alpha \subset \mathbb{C}^n$;*
(3) *$f_\alpha : U_\alpha \to V_\alpha/G_\alpha$ is a homeomorphism;*
(4) *if $U_\alpha \cap U_\beta \neq \emptyset$, and $\pi_s : V_s \to V_s/G_s$ is the natural projection (branched covering) induced by the action of G_s over V_s, then the homeomorphism*

$$f_\beta \circ f_\alpha^{-1} : f_\alpha(U_\alpha \cap U_\beta) \to f_\beta(U_\alpha \cap U_\beta)$$

can be raised to a holomorphic homeomorphism

$$h_{\alpha,\beta} : \pi_\alpha^{-1}(f_\alpha(U_\alpha \cap U_\beta)) \to \pi_\beta^{-1}(f_\beta(U_\alpha \cap U_\beta)).$$

I.3.1. Examples of Complex Orbifold.

(1) An n-dimensional complex manifold M is a particular case of n-dimensional complex orbifold. In this case, $G_\alpha = \{I\}$ and (U_α, f_α) are local charts of M.
(2) Let $X = M$ be an n-dimensional complex manifold and G a group of holomorphic automorphisms that act discontinuously in M. Then the quotient space M/G turns out to be an n-dimensional complex orbifold. If the stabilizers G_x are the trivial group, then we get an n-dimensional complex manifold.

I.3.2. Riemann Orbifold.

Definition 11 (Riemann Orbifold)**.** *A Riemann orbifold $\mathcal{O}$ is a topological Hausdorff space, second countable, such that, each point $p \in S$, there exist:*

(1) *an open $U \subset \mathcal{O}$, $p \in U$;*
(2) *an finite cyclic group G_p, generated by a conformal automorphism of the unitary disk $\mathbb{D}$;*
(3) *a homeomorphism $z : U \to \mathbb{D}/G$; so that if we have two of these homeomorphisms, let's say $z_1 : U_1 \to \mathbb{D}/G_1$ y $z_2 : U_2 \to \mathbb{D}/G_2$ such that $U_1 \cap U_2 \neq \emptyset$, then*

$$z_2 \circ z_1^{-1} : z_1(U_1 \cap U_2) \to z_2(U_1 \cap U_2)$$

can be raised to a holomorphic function (then biholomorphism).

In the above, $x \in X$ with $z(x) = p$ and non-trivial G_p is called a cone point of the orbifold and the order of G_p is called its cone order.

Example 15. Every Riemann surface is a Riemann orbifold. Moreover, every Riemann orbifold has a Riemann surface underlying structure.

Example 16. Let S be a Riemann surface and let G be a group of conformal automorphisms of S. If G acts discontinuously on S, then S/G is a Riemann orbifold (for instance, $S = \mathbb{H}^2$ and G a Fuchsian group). In these cases, the cone points of S/G are exactly the G-classes of those points with non-trivial G-stabilizer.

Example 17 (Holomorphic Galois branched coverings)**.** Let H be a finite group of holomorphic automorphisms of the Riemann surface $\widetilde{S}$ and $p : \widetilde{S} \to S$ a surjective holomorphic function satisfying that $p(x) = p(y)$ if and only if there is $h \in H$ with $h(x) = y$. Then p to be a branched covering, of degree equal to the order of H, with deck group being H. In this case, p is a **holomorphic Galois branched covering**, and the branch points of p are those points whose H-stabilizer is non-trivial (necessarily a cyclic group of order $n > 1$). In addition, we have that if $y \in S$ is a branching value of p, the stabilizers of your pre-images have the same order, let's say the integer $k \geq 2$, so the number of pre-images is $\#\Gamma_i/k$ ([**40**]).

CHAPTER II

Moduli and Torelli spaces of marked surfaces

In this chapter study the moduli space and Torelli space of marked surfaces. We begin with Section 2.1 and 2.2 where we present a summary of existing results about Teichmüller spaces and moduli spaces. In Section 2.3 we present a summary of existing results about about moduli and Teichmüller spaces of marked spheres.

Let S_0 be a compact orientable surface of genus $g \geq 0$ and let $B = \{p_1, \ldots, p_r\} \subset S_0$ be (r) fixed points (B could be an empty set). The surface S_0 is called ***r*-marked surface**, we are interested in the spaces that parameterizes classes of biholomorphisms of Riemann surfaces of genus g with (r) marked points ordered and without order.

II.1. Teichmüller space

Definition 12 (Marking)**.** *A* **marking of** S_0 *is a pair* (S, ϕ)*, where* S *is a closed Riemann surface of genus* g *and* $\phi : S_0 \to S$ *is an orientation-preserving homeomorphism.*

Definition 13 (Equivalence relation between markings)**.** *Two markings* (S_1, ϕ_1) *and* (S_2, ϕ_2) *of* S_0 *is said that are* **equivalent** *if there is a biholomorphism* $T : S_1 \to S_2$ *such that* $\phi_2^{-1} \circ T \circ \phi_1 : S_0 \to S_0$ *fixes each of the points* p_j *and it is homotopic to the identity relative the set* $\{p_1, \ldots, p_r\}$ *(see the figure II.1).*

$$\begin{tikzcd} S_0 \arrow[r, "\phi_1"] \arrow[d, "\phi_2^{-1}\circ T\circ\phi_1"'] \arrow[dr, phantom, "\circlearrowleft"] & S_1 \arrow[d, "T"] \\ S_0 \arrow[r, "\phi_2"] & S_2 \end{tikzcd}$$

FIGURE II.1. Equivalent markings of S_0: ϕ_1 and ϕ_2.

Definition 14 (The Teichmüller space)**. The Teichmüller space of** S_0 **of type (g, r)** $\mathcal{T}_{g,r}$, *is the set of equivalence classes of the above markings. If* $r = 0$ *then* $\mathcal{T}_{g,0} = \mathcal{T}_g$ *is the* **Teichmüller space of genus** g*. We will denote with* $[(S, \phi)]$ *or* $[\phi : S_0 \to S]$ *an element of* $\mathcal{T}_{g,r}$.

Theorem 2. [41] *Let $\mathcal{T}_{g,r}$ be the teichmüller space of S_0 of type (g, r). Then*

(1) If $3g - 3 + r > 0$, then $\mathcal{T}_{g,r}$ is a complex, contractible (it is a ball in a $3g - 3 + r$-dimensional space) manifold of complex dimension $3g - 3 + r$.
(2) If $(g, r) \in \{(1, 0), (0, 4)\}$ then $\mathcal{T}_{g,r} \cong \mathbb{H}$ (where $\mathbb{H}$ is the upper half-plane).
(3) If $(g, r) \in \{(0, 0), (0, 1), (0, 2), (0, 3)\}$ then $\mathcal{T}_{g,r}$ consists of a point.

II.2. The Moduli and Torelli space

Let $\mathrm{Hom}^+(S_0; \{p_1, \ldots, p_r\})$ be the group of orientation-preserving homeomorphisms of S_0 keeping the set $\{p_1, \ldots, p_r\}$ invariant, and let $\mathrm{Hom}^+(S_0; (p_1, \ldots, p_r))$ be its normal subgroup consisting of those orientation-preserving homeomorphisms of S_0 fixing each of the points p_j, $j = 1, \ldots, r$. The subgroup $\mathrm{Hom}_0(S_0; (p_1, \ldots, p_r))$ of $\mathrm{Hom}^+(S_0; (p_1, \ldots, p_r))$ consisting of those being homotopic to the identity relative to the set $\{p_1, \ldots, p_r\}$ is a normal subgroup of $\mathrm{Hom}^+(S_0; \{p_1, \ldots, p_r\})$.

Definition 15 (The modular group and the pure modular group)**.** *The quotient groups*

$$\mathrm{Mod}_{g,[r]} = \mathrm{Hom}^+(S_0; \{p_1, \ldots, p_r\})/\mathrm{Hom}_0(S_0; (p_1, \ldots, p_r)),$$

$$\mathrm{Mod}_{g,r} = \mathrm{Hom}^+(S_0; (p_1, \ldots, p_r))/\mathrm{Hom}_0(S_0; (p_1, \ldots, p_r))$$

are, respectively, the **modular group** *and the* **pure modular group** *of marked surface S_0.*

Note that if $\Psi \in \mathrm{Hom}^+(S_0; \{p_1, \ldots, p_r\})$, then $[\Psi] \in \mathrm{Mod}_{g,[r]}$ contains elements of the form $\Psi \circ \Psi_0$, where $\Psi_0 \in \mathrm{Hom}_0(S_0; (p_1, \ldots, p_r))$.

Let $\Psi \in \mathrm{Hom}^+(S_0; \{p_1, \ldots, p_r\})$, we have that a permutation of the r-points is induced naturally, say $\sigma \in S_r$, with S_r is the symmetric group of r permutations. Therefore we define the surjective homomorphism ρ,

$$\rho : \mathrm{Mod}_{g,[r]} \to S_r : [\Psi] \mapsto \sigma^{-1},$$

which is well defined because it does not depend on the representative of the class, since if we take another representative of the class, $(\Psi \circ \Psi_0) \in [\Psi]$, where $\Psi_0 \in \mathrm{Hom}_0(S_0; (p_1, \ldots, p_r))$, the same σ^{-1} permutation is induced. To verify that ρ is surjective, it suffices to prove that any transposition of the symmetric group S_r has a pre-image, so if $\beta = (a, b)$ is a transposition of S_r, we take the points a and b and enclose them in an open of S_0, which is homeomorphic to a disk of the complex plane $\mathbb{C}$, and we define $\Psi_\beta \in \mathrm{Hom}^+(S_0; \{p_1, \ldots, p_r\})$ being the orientation-preserving homeomorphism acting as (i) the identity outside a disk containg in its interior the two points a, b, (ii) as a rotation of 180 degrees in a smaller disc containing these two points and permuting the points, and (iii) being an homeomorphism in the intermediate annulus (see [47]). So the pre-image of $\beta \in S_r$ is $[\Psi_\beta] \in \mathrm{Mod}_{g,[r]}$. We also have that the kernel of ρ is the pure modular group $\mathrm{Mod}_{g,r}$, therefore we obtain that:

$$S_r \cong \mathrm{Mod}_{g,[r]}/\mathrm{Mod}_{g,r},$$

also called the **Torelli modular group**.

We have a natural action of the group $\mathrm{Mod}_{g,[r]}$ in the Teichmüller space $\mathcal{T}_{g,r}$ defined by:

$$\alpha : \left\{ \begin{array}{ccc} \mathrm{Mod}_{g,[r]} \times \mathcal{T}_{g,r} & \to & \mathcal{T}_{g,r} \\ ([\Psi], [\phi : S_0 \to S]) & \mapsto & [\phi \circ \Psi^{-1} : S_0 \to S] \end{array} \right. .$$

Proposition 1. *The above does not depend on the representatives of the classes and it is an action.*

Proof. First we take other representatives of the classes $[\Psi] \in \mathrm{Mod}_{g,[r]}$ and , $[\phi : S_0 \to S] \in \mathcal{T}_{g,r}$ and we'll show that your alpha image gives us the class $[\phi \circ \Psi^{-1} : S_0 \to S]$. Let $\Psi_1 = \Psi \circ \Psi_0$ with $\Psi_0 \in \mathrm{Hom}_0(S_0; (p_1, \ldots, p_r))$, and $\phi_1 : S_0 \to S_1$ equivalent to ϕ, i.e., $\phi_1 = T \circ \phi \circ \eta^{-1}$ where $T : S \to S_1$ is a biholomorphism and $\eta \in \mathrm{Hom}_0(S_0; (p_1, \ldots, p_r))$. We will verify that $(\phi_1 \circ \Psi_1^{-1}) \sim (\phi \circ \Psi^{-1})$. So, as $(\phi_1 \circ \Psi_1^{-1}) = T \circ \phi \circ \eta^{-1} \circ \Psi_0^{-1} \circ \Psi^{-1}$ we have the following diagram:

$$\begin{array}{ccccc} S_0 & \xrightarrow{\Psi_1^{-1}} & S_0 & \xrightarrow{\phi_1} & S_1 \\ {\scriptstyle f}\downarrow & & & & \downarrow{\scriptstyle T^{-1}} \\ S_0 & \xrightarrow{\Psi^{-1}} & S_0 & \xrightarrow{\phi} & S \end{array}$$

Figure II.2. Equivalent markings: $\phi \circ \Psi^{-1}$ and $\phi_1 \circ \Psi_1^{-1}$

Where $f = (\Psi \circ (\eta^{-1} \circ \Psi_0^{-1}) \circ \Psi^{-1}) \in \mathrm{Hom}_0(S_0; (p_1, \ldots, p_r))$, with this we have to, $(\phi_1 \circ \Psi_1^{-1}) \sim (\phi \circ \Psi^{-1})$.

Now we will show that α defines an action, that is, let $[\Psi], [\tilde{\Psi}] \in \mathrm{Mod}_{g,[r]}$, and $[\phi : S_0 \to S] \in \mathcal{T}_{g,r}$, we will verify that:

$$\alpha([\Psi] * [\tilde{\Psi}], [\phi]) = \alpha([\Psi], \alpha([\tilde{\Psi}], [\phi])),$$

where the operation $*$ is induced by the composition, so, we have:

$$\alpha([\Psi] * [\tilde{\Psi}], [\phi]) = \alpha([\Psi \circ \tilde{\Psi}], [\phi]) = [\phi \circ (\Psi \circ \tilde{\Psi})^{-1}] = [\phi \circ \tilde{\Psi}^{-1} \circ \Psi^{-1}],$$

on the other hand,

$$\alpha([\Psi], \alpha([\tilde{\Psi}], [\phi])) = \alpha([\Psi], [\phi \circ \tilde{\Psi}^{-1}]) = [\phi \circ \tilde{\Psi}^{-1} \circ \Psi^{-1}].$$

□

Theorem 3 ([41]). *The modular group* $\mathrm{Mod}_{g,[r]}$ *(in particular, the pure modular group* $\mathrm{Mod}_{g,r}$*) acts properly discontinuously on* $\mathcal{T}_{g,r}$ *as a group of holomorphic automorphisms.*

In particular, the stabilizer of each point by $\mathrm{Mod}_{g,[r]}$ *is finite, and the quotient spaces* $\mathcal{T}_{g,r}/\mathrm{Mod}_{g,[r]}$ *and* $\mathcal{T}_{g,r}/\mathrm{Mod}_{g,r}$ *are both complex orbifolds of the same dimension that* $\mathcal{T}_{g,r}$*.*

Theorem 4 ([46], [25])**.** *For (i)* $r = 0$ *and* $g \geq 2$ *and for (ii)* $r \geq 1$ *and* $2g + r > 4$*, the modular group* $\mathrm{Mod}_{g,[r]}$ *is the* **full group of holomorphic automorphisms** *of* $\mathcal{T}_{g,r}$*.*

Let $[\phi : S_0 \to S]$ be a class marking in the Teichmüller space $\mathcal{T}_{g,r}$, we will denote by $[[\phi : S_0 \to S]]_{\mathrm{Mod}_{g,r}}$ or $[[\phi]]_{\mathrm{Mod}_{g,r}}$ the class in the quotient space $\mathcal{T}_{g,r}/\mathrm{Mod}_{g,r}$.

Proposition 2. *Let* $[\phi_1 : S_0 \to S_1]$ *and* $[\phi_2 : S_0 \to S_2]$ *be class of markings in the Teichmüller space* $\mathcal{T}_{g,r}$*, they are* $\mathrm{Mod}_{g,r}$*-equivalents if and only if there is a biholomorphism* $h : S_1 \to S_2$ *such that* $h(\phi_1(p_j)) = \phi_2(p_j)$*, for* $j = 1$ *to* r*.*

Proof. Let $[\phi_1 : S_0 \to S_1], [\phi_2 : S_0 \to S_2] \in \mathcal{T}_{g,r}$, let's notice with $q_j := \phi_1(p_j)$ and with $m_j := \phi_2(p_j)$ for $j = 1$ to r, to the marked points of S_1 and S_2 respectively. First we show that if $[\phi_1 : S_0 \to S_1]$ and $[\phi_2 : S_0 \to S_2]$ belong to $[[\phi_1 : S_0 \to S_1]]_{\mathrm{Mod}_{g,r}}$, then S_1 and S_2 are biholomorphic-equivalents. Since $[\phi_1 : S_0 \to S_1]$ and $[\phi_2 : S_0 \to S_2]$ are $\mathrm{Mod}_{g,r}$-equivalents, then there is $\tilde{\Psi} \in \mathrm{Hom}^+(S_0, (p_1, \dots, p_r))$ such that,

$$\alpha([\tilde{\Psi}], [\phi_1 : S_0 \to S_1]) = [\phi_1 \circ \tilde{\Psi}^{-1} : S_0 \to S_1] = [\phi_2 : S_0 \to S_2],$$

therefore $(\phi_1 \circ \tilde{\Psi}^{-1}) \sim \phi_2$, with this there is a biholomorphism $h : S_1 \to S_2$ such that $h(q_j) = m_j$, for $j = 1$ to r. In the other sense, suppose that there is a biholomorphism $h : S_1 \to S_2$ such that $h(q_j) = m_j$, for $j = 1$ to r, we will show that $[\phi_1 : S_0 \to S_1]$ and $[\phi_2 : S_0 \to S_2]$ belong to $[[\phi_1 : S_0 \to S_1]]_{\mathrm{Mod}_{g,r}}$, i.e., there is $[\tilde{\Psi}] \in \mathrm{Mod}_{g,r}$ such that $[\phi_1 \circ \tilde{\Psi}^{-1} : S_0 \to S_1] = [\phi_2 : S_0 \to S_2]$, so we define $\tilde{\Psi} := \phi_2^{-1} \circ h \circ \phi_1$, with this it is verified that $(\phi_1 \circ \tilde{\Psi}^{-1}) \sim \phi_2$.

□

Definition 16 (Ordered moduli space of type (g, r) or Torelli space of r-marked surfaces of genus g)**.** *The* **ordered moduli space of type** (g, r)*, also called the* **Torelli space of** r**-marked surfaces of genus** g *is the quotient complex orbifold*

$$\mathcal{M}_{g,r} := \mathcal{T}_{g,r}/\mathrm{Mod}_{g,r}$$

of dimension $3g - 3 + r$*. This space consists of classes of biholomorphisms of marked surfaces of genus g with r ordered marked points.*

Definition 17 (Unordered Moduli space of type (g, r) or Moduli space of type (g, r))**.** *The* **unordered Moduli space of type** (g, r) *is the quotient complex orbifold*

$$\mathcal{M}_{g,[r]} := \mathcal{T}_{g,r}/\mathrm{Mod}_{g,[r]},$$

of dimension $3g-3+r$. This space consists of classes of biholomorphisms of marked surfaces of genus g with r marked points without order.

As a consequence of Theorem 4 we obtain the following.

Corollary 1. *For $r \geq 1$ and $2g+r>4$, the Torelli modular group, $S_r \cong \mathrm{Mod}_{g,[r]}/\mathrm{Mod}_{g,r}$, acts properly discontinuously on $\mathcal{M}_{g,r}$ as a group of holomorphic automorphisms with quotient orbifold $\mathcal{M}_{g,[r]}$. This action corresponds to the permutations of the r marked points.*

From the above, we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{T}_{g,r} \arrow[rr, "\pi_{g,r}"] \arrow[d, "\pi_{g,[r]}"'] & & \mathcal{M}_{g,r} = \mathcal{T}_{g,r}/\mathrm{Mod}_{g,r} \arrow[dll, "P_{g,r}"] \\
\mathcal{M}_{g,[r]} = \mathcal{T}_{g,r}/\mathrm{Mod}_{g,[r]} = \mathcal{M}_{g,r}/S_r & &
\end{tikzcd}$$

FIGURE II.3. Modular groups actions of $\mathrm{Mod}_{g,[r]}$ and $\mathrm{Mod}_{g,r}$

where $\pi_{g,[r]} : \mathcal{T}_{g,r} \to \mathcal{T}_{g,r}/\mathrm{Mod}_{g,[r]}$, $\pi_{g,r} : \mathcal{T}_{g,r} \to \mathcal{T}_{g,r}/\mathrm{Mod}_{g,r}$, and $P_{g,r} : \mathcal{M}_{g,r} \to \mathcal{M}_{g,r}/S_r$ are respectively the natural projections of the action of groups $\mathrm{Mod}_{g,[r]}$ and $\mathrm{Mod}_{g,r}$ on the space $\mathcal{T}_{g,r}$, and the action of group $S_r = \mathrm{Mod}_{g,[r]}/\mathrm{Mod}_{g,r}$ on the space $\mathcal{M}_{g,r}$.

II.3. The Moduli $\mathcal{M}_{0,[n+1]}$ and Torelli $\mathcal{M}_{0,n+1}$ spaces of marked spheres

From here on we will consider $r = n+1$ with $n \geq 3$, and the genus $g = 0$, therefore we will work with the Riemann spheres marked in $n+1$ points, with $\{\infty, 0, 1\} \subset \{p_1, \ldots, p_{n+1}\}$. As well as, consider the Teichmüller space of type $(0, n+1)$, $\mathcal{T}_{0,n+1}$, the modular group $\mathrm{Mod}_{0,[n+1]}$, the pure modular group $\mathrm{Mod}_{0,n+1}$, the Torelli modular group $S_{n+1} \cong \mathrm{Mod}_{0,[n+1]}/\mathrm{Mod}_{0,n+1}$, the Torelli space $\mathcal{M}_{0,n+1}$ and the Moduli space $\mathcal{M}_{0,[n+1]}$.

It is known that, for $n \geq 4$, $\mathrm{Mod}_{0,n+1}$ acts freely on $\mathcal{T}_{0,n+1}$, so the Torelli space $\mathcal{M}_{0,n+1}$ is a complex manifold of complex dimension $n-2$. For $n = 3$ the group $\mathrm{Mod}_{0,4}$ acts non-freely on $\mathcal{T}_{0,4}$ (in fact, every point in $\mathcal{T}_{0,4}$ has non-trivial stabilizer).

Remark 4. Igusa [**35**] observed, by using the invariants of the binary sextics, that $\mathcal{M}_{0,[6]}$ can be seen as the quotient of $\mathbb{C}^3$ by the action of the cyclic group of order five $\langle (x,y,z) \mapsto (\omega_5 x, \omega_5^2 y, \omega_5^3 z) \rangle$, where $\omega_5 = e^{2\pi i/5}$. Using invariants of binary quintics, it can also be obtained that $\mathcal{M}_{0,[5]}$ is the quotient of $\mathbb{C}^2$ by the cyclic group of order two $\langle (x,y) \mapsto (-x,-y) \rangle$.

In [**43**] it was observed that, for $n \geq 6$, the moduli space $\mathcal{M}_{0,[n+1]}$ cannot be seen as the quotient of $\mathbb{C}^{n-2}$ by the action of a finite linear group.

II.3.1. Models for the Moduli and Torelli spaces of marked spheres. Next, we proceed to recall a natural model $\Omega_n \subset \mathbb{C}^{n-2}$ for the Torelli space $\mathcal{M}_{0,n+1}$ together with an explicit form of its automorphisms (see, for instance, [**43**]).

Let $\mathcal{X}_n \subset \widehat{\mathbb{C}}^{n+1}$ be the configuration space of ordered $(n+1)$-tuples whose coordinates are pairwise different. Two tuples $(p_1, \ldots, p_{n+1}), (q_1, \ldots, q_{n+1}) \in \mathcal{X}_n$ are equivalent if there is a Möbius transformation $M \in \mathrm{PSL}_2(\mathbb{C})$ such that $M(p_j) = q_j$, for $j = 1, \ldots, n+1$. As for $(p_1, \ldots, p_{n+1}) \in \mathcal{X}_n$, there is a (unique) Möbius transformation M such that $M(p_1) = \infty$, $M(p_2) = 0$ and $M(p_3) = 1$, each $(p_1, \ldots, p_{n+1})$ is equivalent to a unique one of the form $(\infty, 0, 1, \lambda_1, \ldots, \lambda_{n-2})$. It follows that the quotient space $\mathcal{X}_n/\mathrm{PSL}_2(\mathbb{C})$ can be identified with

$$\Omega_n = \{(z_1, \ldots, z_{n-2}) : z_j \in \mathbb{C} \setminus \{0, 1\},\ z_i \neq z_j\} \subset \mathbb{C}^{n-2}.$$

By the uniformization theorem, each point of $\mathcal{T}_{0,n+1}$ is the class of a pair of the form $(\widehat{\mathbb{C}}, \phi)$, where $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is an orientation-preserving homeomorphism that fixes $\infty, 0, 1$. This is given, because if $\tilde{\phi} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a marking of $\widehat{\mathbb{C}}$, the application $T \circ \tilde{\phi}$ is a marking equivalent to $\tilde{\phi}$ that fixes $\infty, 0, 1$, where T is the only Mobiüs transformation that sends $\tilde{\phi}(\infty)$ to ∞, $\tilde{\phi}(0)$ to 0 and $\tilde{\phi}(1)$ to 1.

The following bijective application is defined:

$$f : \left\{ \begin{array}{ccc} \mathcal{T}_{0,n+1}/\mathrm{Mod}_{0,n+1} & \to & \Omega_n \\ [[\phi]]_{\mathrm{Mod}_{0,n+1}} & \mapsto & (\lambda_1, \ldots, \lambda_{n-2}) \end{array} \right. ,$$

where $\phi(\infty) = \infty, \phi(0) = 0, \phi(1) = 1, \phi(p_4) = \lambda_1, ..., \phi(p_{n+1}) = \lambda_{n-2}$. The application is well defined because if we take another representative $[\tilde{\phi}] \in [[\phi]]_{\mathrm{Mod}_{0,n+1}}$, where $\tilde{\phi}$ fixes $\infty, 0$, and 1, then we have that, there is $[\Psi] \in \mathrm{Mod}_{0,n+1}$ such that, $(\tilde{\phi} \circ \Psi^{-1}) \sim \phi$, with this we have $\phi(p_j) = (\tilde{\phi} \circ \Psi^{-1})(p_j) = \tilde{\phi}(p_j)$, therefore to $\left[[\tilde{\phi}]\right]_{\mathrm{Mod}_{0,n+1}}$ corresponds to the same tuple $(\lambda_1, \ldots, \lambda_{n-2})$. The injectivity is given because, if we take $[[\phi_1]]_{\mathrm{Mod}_{0,n+1}}, [[\phi_2]]_{\mathrm{Mod}_{0,n+1}}$ in $\mathcal{T}_{0,n+1}/\mathrm{Mod}_{0,n+1}$, such that $f([[\phi_1]]_{\mathrm{Mod}_{0,n+1}}) = f([[\phi_2]]_{\mathrm{Mod}_{0,n+1}})$, that is: $\phi_1(p_j) = \phi_2(p_j)$, for $j = 1$ to $n+1$, if we take $\Psi := (\phi_2^{-1} \circ \phi_1) \in \mathrm{Hom}^+(\widehat{\mathbb{C}}; (p_1, \ldots, p_{n+1}))$, then $[[\phi_1]]_{\mathrm{Mod}_{0,n+1}} = [[\phi_2]]_{\mathrm{Mod}_{0,n+1}}$. The application is surjective because, let be $(\lambda_1, \ldots, \lambda_{n-2}) \in \Omega_n$, then you can build a homeomorphism ϕ that preserves the orientation of the Riemann sphere that sends the tuple $(p_1, \ldots, p_{n+1})$ in $(\infty, 0, 1, \lambda_1, \ldots, \lambda_{n-2})$. So we may identify $\mathcal{M}_{0,n+1}$ with Ω_n.

The permutation action of $\mathfrak{S}_{n+1}$ on the coordinates of the tuples $(p_1, \ldots, p_{n+1}) \in \mathcal{X}_n$ is transported to the action of a group $\mathbb{G}_n$ of holomorphic automorphisms of Ω_n, as describe below. Let us fix some $\sigma \in \mathfrak{S}_{n+1}$. Each point $\lambda := (z_1, \ldots, z_{n-2}) \in \Omega_n$ corresponds to the ordered tuple $(p_1 = \infty, p_2 = 0, p_3 = 1, p_4 = z_1, \ldots, p_{n+1} = z_{n-2}) \in \mathcal{X}_n$. We now consider the new tuple $(p_{\sigma^{-1}(1)}, \ldots, p_{\sigma^{-1}(n+1)}) \in \mathcal{X}_n$. There is a unique Möbius transformation $M_{\sigma,\lambda}$

such that $M_{\sigma,\lambda}(p_{\sigma^{-1}(1)}) = \infty$, $M_{\sigma,\lambda}(p_{\sigma^{-1}(2)}) = 0$ and $M_{\sigma,\lambda}(p_{\sigma^{-1}(3)}) = 1$; this given as

$$M_{\sigma,\lambda}(x) = \frac{(x - p_{\sigma^{-1}(2)})(p_{\sigma^{-1}(3)} - p_{\sigma^{-1}(1)})}{(x - p_{\sigma^{-1}(1)})(p_{\sigma^{-1}(3)} - p_{\sigma^{-1}(2)})}.$$

As $\left(M_{\sigma,\lambda}(p_{\sigma^{-1}(4)}), \dots, M_{\sigma,\lambda}(p_{\sigma^{-1}(n+1)})\right) \in \Omega_n$, the map

$$T_\sigma : \Omega_n \to \Omega_n : \lambda = (z_1, \dots, z_{n-2}) \mapsto \left(M_{\sigma,\lambda}(p_{\sigma^{-1}(4)}), \dots, M_{\sigma,\lambda}(p_{\sigma^{-1}(n+1)})\right),$$

is an holomorphic automorphism of Ω_n. This procedure provides of a surjective homomorphism (we are using multiplication of permutations from the left)

$$\Theta_n : \left\{ \begin{array}{ccc} \mathfrak{S}_{n+1} & \to & \mathbb{G}_n = \langle A, B \rangle \\ \sigma & \mapsto & \Theta_n(\sigma) := T_\sigma \end{array} \right. ,$$

where $A = \Theta_n((1,2))$ and $B = \Theta_n((1,2,\dots,n+1))$. It can be checked that

$$A\,(z_1, \dots, z_{n-2}) = \left(\frac{1}{z_1}, \dots, \frac{1}{z_{n-2}}\right),\ B\,(z_1, \dots, z_{n-2}) = \left(\frac{z_{n-2}}{z_{n-2} - 1}, \frac{z_{n-2}}{z_{n-2} - z_1}, \dots, \frac{z_{n-2}}{z_{n-2} - z_{n-3}}\right),$$

If $n = 3$, then $K_3 := \ker(\Theta_3) = \{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\} \cong C_2^2$; so $\mathfrak{S}_4/C_2^2 \cong \mathfrak{S}_3 \cong \mathbb{G}_3$. For $n \geq 4$ the kernel of Θ_n is just the trivial group; so $\mathbb{G}_n \cong_{\Theta_n} \mathfrak{S}_{n+1}$.

Next we will see in an example the previous procedure to define Θ_n.

Example 18. If $n = 3$, we have $\Omega_3 = \mathbb{C} \setminus \{0,1\}$, and $\Theta_3 : \mathfrak{S}_4 \to \mathbb{G}_3 : \sigma \mapsto \Theta_3(\sigma) := T_\sigma$. Let $\lambda \in \Omega_3$, and $\sigma = (1,2) \in \mathfrak{S}_4$, we have the tuple $(p_1 = \infty, p_2 = 0, p_3 = 1, p_4 = \lambda)$, we get a new tuple $(p_{\sigma^{-1}(1)} = 0, p_{\sigma^{-1}(2)} = \infty, p_{\sigma^{-1}(3)} = 1, p_{\sigma^{-1}(4)} = \lambda)$, by the action of σ^{-1} over first tuple, subsequently we apply the only Möbius transformation that sends zero to infinity, infinity to zero, and one to one, it is $M_{\sigma,\lambda}(x) = 1/x$, so, we have the new tuple $(\infty, 0, 1, 1/\lambda)$ and we define $T_\sigma(\lambda) = 1/\lambda$ (see the figure II.4).

$$\begin{array}{l} (\infty, 0, 1, \lambda) \\ \quad \Big\downarrow {\scriptstyle \sigma^{-1}=(1,2)} \\ (0, \infty, 1, \lambda) \\ \quad \Big\downarrow {\scriptstyle M_{\sigma,\lambda}(x)=1/x} \\ (\infty, 0, 1, 1/\lambda) \qquad \Rightarrow \qquad T_\sigma(\lambda) = 1/\lambda. \end{array}$$

FIGURE II.4. Diagram of the procedure to obtain T_σ

Continuing with this procedure, we obtain for each $\sigma \in \mathfrak{S}_4$ the following values of T_σ: $\sigma_1 = e \mapsto T_{\sigma_1}(\lambda) = \lambda$, $\sigma_2 = (1,2) \mapsto T_{\sigma_2}(\lambda) = 1/\lambda$, $\sigma_3 = (1,3) \mapsto T_{\sigma_3}(\lambda) = \lambda/(\lambda - 1)$, $\sigma_4 = (2,3) \mapsto T_{\sigma_4}(\lambda) = 1 - \lambda$, $\sigma_5 = (1,3,2) \mapsto T_{\sigma_5}(\lambda) = (\lambda - 1)/\lambda$ and $\sigma_6 = (1,2,3) \mapsto$

$T_{\sigma_6}(\lambda) = 1/(1-\lambda)$, all these permutations in particular, are in $\mathfrak{S}_3$, the permutations that are in $\ker(\Theta_3)$ correspond to T_e and for the other elements of $\mathfrak{S}_4$ the values of T_σ are repeated as we saw earlier $\mathfrak{S}_4/\ker(\Theta_3) = \mathfrak{S}_4/C_2^2 \cong \mathfrak{S}_3 \cong \mathbb{G}_3$.

We have that $\mathbb{G}_n$ **is the full group of holomorphic automorphisms of** Ω_n. For $n = 3$ this is trivial. For $n \geq 4$, this is as follows: Since the Teichmüller space (which is contractible) is the universal covering of Ω_n. If α is a holomorphic automorphism of Ω_n, then it we lift to a holomorphic automorphism of the Teichmuller space, that is, α belongs to the modular group by theorem 4 , for $n \geq 4$. Therefore $\mathbb{G}_n$ is the full group of automorphisms of Ω_n. So, every anti-holomorphic automorphism of Ω_n has the form $T \circ J$, where $T \in \mathbb{G}_n$ and $J(z_1, \ldots, z_{n-2}) = (\overline{z}_1, \ldots, \overline{z}_{n-2})$ (which is induced by complex conjugation).

Definition 18 (Symmetries of Ω_n)**.** *The* **symmetries of** Ω_n *are those anti-holomorphic automorphisms of order two. In particular, J is a symmetry. As J commutes with every element of $\mathbb{G}_n$, the symmetries of Ω_n are those of the form $T \circ J$, where $T^2 = I$.*

Summarizing all the above is the following.

Lemma 1. *Let $n \geq 3$, $\mathbb{G}_n = \langle A, B \rangle$ and $\Theta_n : \mathfrak{S}_{n+1} \to \mathbb{G}_n : \sigma \mapsto T_\sigma$ be the surjective homomorphism as defined above. Then*

(1) $\ker(\Theta_3) = \{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\} \cong C_2^2$ *and $\mathbb{G}_3 \cong \mathfrak{S}_3$ is the full group of holomorphic automorphisms of Ω_3;*
(2) if $n \geq 4$, $\mathbb{G}_n \cong_{\Theta_n} \mathfrak{S}_{n+1}$ is the full group of holomorphic automorphisms of Ω_n.
(3) The anti-holomorphic automorphisms of Ω_n are those of the form $T \circ J$, where $T \in \mathbb{G}_n$. Those of order two are for which $T^2 = I$.
(4) The quotient orbifold $\Omega_n/\mathbb{G}_n$ is biholomorphic to the moduli space $\mathcal{M}_{0,[n+1]}$.

In the rest of document we use the model $\mathcal{M}_{0,n+1} = \Omega_n$ and we fix a regular branched cover $\pi_n : \Omega_n \to \Omega_n/\mathbb{G}_n$ with deck group $\mathbb{G}_n$, and therefore we also consider the model $\mathcal{M}_{0,[n+1]} = \Omega_n/\mathbb{G}_n$.

II.3.2. The branch locus and real locus of the $\mathcal{M}_{0,[n+1]}$. We are interested in studying the connectivity of certain special subsets of $\mathcal{M}_{0,[n+1]}$, which we will define below.

Definition 19 (The branch locus of the $\mathcal{M}_{0,[n+1]}$)**.** *Let $\mathcal{M}_{0,[n+1]}$ be the Moduli space of isomorphism classes of marked spheres in (n+1) points. Its* **branch locus** $\mathcal{B}_{0,[n+1]} \subset \mathcal{M}_{0,[n+1]}$ *consists of the isomorphism classes of those $(n+1)$-marked spheres admitting non-trivial holomorphic automorphisms.*

Remark 5. According to the models of $\mathcal{M}_{0,n+1} = \Omega_n$ and $\mathcal{M}_{0,[n+1]} = \Omega_n/\mathbb{G}_n$, the previous definition indicates that the branch locus $\mathcal{B}_{0,[n+1]} \subset \Omega_n/\mathbb{G}_n$ consists of the images under π_n of those points of Ω_n with non-trivial $\mathbb{G}_n$-stabilizer, i.e.,

$$\begin{array}{rcl}\mathcal{B}_{0,[n+1]} & = & \{\pi_n(\lambda)/\lambda \in \Omega_n, \mathrm{Stab}_{\mathbb{G}_n}(\lambda) \neq \{I : z \mapsto z\}\} \\ & = & \bigcup_{T\in\mathbb{G}_n\setminus\{I\}} \pi_n(Fix(T)),\end{array}$$

where $\mathrm{Fix}(T)$ is the set of fixed points of T on Ω_n.

The complex orbifold $\mathcal{M}_{0,[n+1]}$ also admits a natural anti-holomorphic automorphism $\widehat{J}$ of order two (a real structure), this being induced by the complex conjugation on the Riemann sphere, see the commutative diagram:

$$\begin{tikzcd} \Omega_n \arrow[r, "J"] \arrow[d, "\pi_n"'] \arrow[dr, phantom, "\circlearrowleft"] & \Omega_n \arrow[d, "\pi_n"] \\ \Omega_n/\mathbb{G}_n \arrow[r, "\widehat{J}"] & \Omega_n/\mathbb{G}_n \end{tikzcd}$$

FIGURE II.5. Real structure $\hat{J}$

The fixed points of the real structure $\widehat{J}$ are called the real points. These points are the projections of those points $\lambda \in \Omega_n$ such that λ and $J(\lambda) = \overline{\lambda} \in \Omega_n$ are $\mathbb{G}_n$-equivalents. This is equivalent to have some $T \in \mathbb{G}_n$ such that $T(\lambda) = \overline{\lambda}$, that is, if λ is a fixed point of $\widehat{T} = T \circ J$, which is an anti-holomorphic automorphism of Ω_n. So, we have the following definition.

Definition 20 (The real locus of $\mathcal{M}_{0,[n+1]}$). *The locus $\mathcal{M}_{0,[n+1]}(\mathbb{R})$ of the fixed points of the real structure of $\mathcal{M}_{0,[n+1]}$ (the real points) consists of the isomorphism classes of those marked spheres admitting an anti-holomorphic automorphism. Inside this locus is the* **real locus** $\mathcal{M}^{\mathbb{R}}_{0,[n+1]}$, *consisting of those classes of marked spheres admitting an anti-holomorphic involution.*

Remark 6. We have that,

$$\mathcal{M}^{\mathbb{R}}_{0,[n+1]} = \bigcup \pi_n(\mathrm{Fix}(S)),$$

where S runs over all the symmetries of Ω_n and $\mathrm{Fix}(S)$ is the set of fixed points of S.

Remark 7. A point $\lambda = (\lambda_1, \dots, \lambda_{n-2}) \in \Omega_n$ is fixed by an anti-holomorphic automorphism if and only if the collection $\{\infty, 0, 1, \lambda_1, \dots, \lambda_{n-2}\}$ is invariant under an anti-holomorphic automorphism of the Riemann sphere (that is, an extended Möbius transformation) which is of the same order. (1) If n is even, then an anti-holomorphic automorphism keeping invariant a collection of $n+1$ points must be a reflection (each orbit under the action of an anti-holomorphic automorphism of order $2m$, where $m \geq 2$, is of even cardinality), so $\mathcal{M}_{0,[n+1]}(\mathbb{R}) = \mathcal{M}^{\mathbb{R}}_{0,[n+1]}$. (2) If n is odd, then in general $\mathcal{M}_{0,[n+1]}(\mathbb{R})$ and $\mathcal{M}^{\mathbb{R}}_{0,[n+1]}$ are different (see example 19 for $n = 9$). In this case, if the above collection of points is invariant under some anti-holomorphic automorphism of order $2m$, where $m \geq 2$, then we may assume (up to conjugation by a suitable Möbius transformation) that such an automorphism is given by $T(z) = e^{\pi i/m}/\overline{z}$. It is possible to move all the points continuously to the unit circle, keeping the invariance under T. This, in particular, asserts that every connected component of $\mathcal{M}_{0,[n+1]}(\mathbb{R})$ intersects the real locus $\mathcal{M}^{\mathbb{R}}_{0,[n+1]}$.

Example 19. Let $n = 9$, suppose we have a sphere marked in 10 points, given by the collection

$$C = \left\{\infty, 0, 1, -1, i, -i, \lambda := 2+i, -\lambda = -2-i, \frac{i}{\overline{\lambda}} = -\frac{1}{5} + \frac{2i}{5}, -\frac{i}{\overline{\lambda}} = \frac{1}{5} - \frac{2i}{5}\right\},$$

the extended Möbius transformations $\tau(x) = \dfrac{i}{\overline{x}}$ of order four leaves invariant the collection of points, that is, the sphere admits a anti-holormorphic automorphism. Also we have that $\tau^2(x) = -x$ is a Möbius transformation. This marked sphere does not admit an anti-holomorphic automorphism of order two because, if there is a η involution that leaves the collection C invariant then $\eta \circ \tau$ must leave this collection invariant, and also $\eta \circ \tau = M$ is a Möbius transformation distinct than τ^2, therefore we must find a Möbius transformation M such that $M(C) = C$. If we see the collection in the extended complex plane,$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the points $\{\infty, 0, 1, -1\}$ and the points $\{1, -1, i, -i\}$ are in two circles, besides the points in the sub-collection

$$P = \left\{\lambda := 2+i, -\lambda = -2-i, \frac{i}{\overline{\lambda}} = \frac{-1}{5} + \frac{2i}{5}, \frac{-i}{\overline{\lambda}} = \frac{1}{5} + \frac{-2i}{5}\right\}$$

are not in a circle, so Möbius transformations M must permute the elements of the circles or take the one circle in the other, for so for the rest of the sub-collection P the only option is that M permute between them. It is verified that the Möbius transformations that leave invariant the sub-collection P do not leave invariant all the collection C or in its defect $M = \tau^2$, which leads to a contradiction, verifying that the sphere does not admit an involution. Thus, we observe that in general $\mathcal{M}_{0,[n+1]}(\mathbb{R})$ and $\mathcal{M}^{\mathbb{R}}_{0,[n+1]}$ are different.

As initially commented, we are interested in studying the connectivity of the branch locus and the real locus, here we will describe some known results.

In [**5**] it was proved that $\mathcal{B}_{g,[0]} \subset \mathcal{M}_{g,[0]} = \mathcal{M}_g$ is connected only for $g \in \{3, 4, 13, 17, 19, 59\}$. In [**20**] it was proved that $\mathcal{M}_{g,[0]}(\mathbb{R}) \subset \mathcal{M}_g$ is also connected but that $\mathcal{M}_{g,[0]}(\mathbb{R}) \setminus \mathcal{M}^{\mathbb{R}}_{g,[0]}$ is not in general connected.

If $n = 3$, then $\mathcal{M}_{0,4}$ can be identified with $\Omega_3 = \mathbb{C} \setminus \{0,1\}$. In this case, $\mathbb{G}_3 \cong \mathfrak{S}_3$ (the action of $\mathfrak{S}_4$ on $\mathcal{M}_{0,4}$ is not faithful as it contains a normal subgroup $K_3 \cong C_2^2$ acting trivially). In particular, $\mathcal{B}_{0,[4]} = \mathcal{M}_{0,[4]}$. The quotient orbifold $\mathcal{M}_{0,[4]} = \Omega_3/\mathbb{G}_3$ can be identified with the complex plane $\mathbb{C}$ with two cone points, one of order two and the other of order three, (the two cone points corresponds exactly to those 4-marked spheres whose of conformal automorphisms is bigger than C_2^2), this is given by considering the classical Klein modular j-function $j = 4(\lambda^2 - \lambda + 1)^3/(27\lambda^2(\lambda-1)^2)$, which is a regular branched cover with deck group $\mathbb{G}_3 \cong \mathfrak{S}_3$ (see [**41**]). Also, $\mathcal{M}_{0,[4]}^{\mathbb{R}} = \mathbb{R}$ (see the figure II.6).

If $n \geq 4$, then $\mathcal{M}_{0,n+1}$ can be identified with the domain $\Omega_n \subset \mathbb{C}^{n-2}$. In this case, $\mathbb{G}_n \cong \mathfrak{S}_{n+1}$ acts faithfully as the full group of holomorphic automorphisms of $\mathcal{M}_{0,n+1}$ [**46**, **25**] and $\Omega_n/\mathbb{G}_n = \mathcal{M}_{0,[n+1]}$. If $\mathrm{Sing}_{0,[n+1]} \subset \mathcal{M}_{0,[n+1]}$ is the locus of non-manifold points, then: (i) for $n \geq 6$, $\mathrm{Sing}_{0,[n+1]} = \mathcal{B}_{0,[n+1]}$ [**43**] and (ii) for $n \in \{4,5\}$, the singular locus consists of exactly one point [**35**]. If, for $T \in \mathbb{G}_n \setminus \{I\}$, we denote by $\mathrm{Fix}(T) \subset \Omega_n$ the locus of its fixed points, then in [**47**] it was observed that, for $\mathrm{Fix}(T) \neq \emptyset$ (which might not be connected), its projection to $\mathcal{M}_{0,[n+1]}$ is connected. In Section III.1 we study the connectivity of $\mathcal{B}_{0,[n+1]}$ (see Theorem 6).

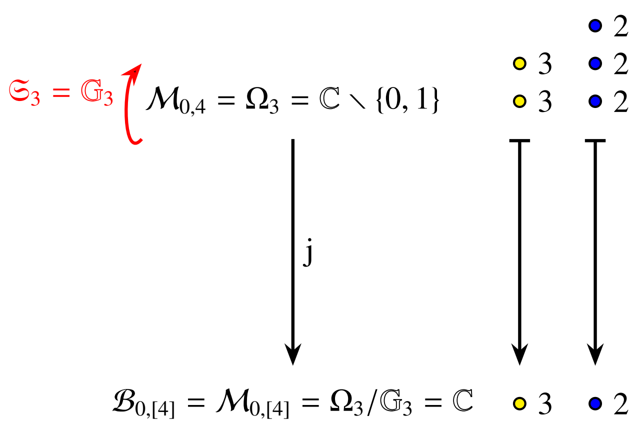


FIGURE II.6. Branch locus for n=3

CHAPTER III

Results

III.1. The connectivity of the Branch locus

From now on, we assume $n \geq 4$. The branch locus $\mathcal{B}_{0,[n+1]} \subset \Omega_n/\mathbb{G}_n$ consists of the images under π_n of those points with non-trivial $\mathbb{G}_n$-stabilizer. For $\lambda = (\lambda_1, \ldots, \lambda_{n-2}) \in \Omega_n$, we set

$$C_\lambda = \{p_1 = \infty, p_2 = 0, p_3 = 1, p_4 = \lambda_1, \ldots, p_{n+1} = \lambda_{n-2}\},$$

we denote by G_λ^+ be the group of Möbius transformations keeping invariant the set C_λ, and by G_λ the group generated by G_λ^+ and those extended Möbius transformations (compositions of complex conjugation with a Möbius transformation) keeping invariant C_λ (so either $G_\lambda = G_\lambda^+$ or $[G_\lambda : G_\lambda^+] = 2$). As the cardinality of C_λ is bigger than three, it follows that G_λ is finite. For the generic case, G_λ is trivial and in the non-generic case (that is, when we place them properly), G_λ^+ is isomorphic to a finite group of Möbius transformations, which can be a cyclic group C_m, a dihedral group D_m (of order $2m$), an alternating group $\mathcal{A}_4$ or $\mathcal{A}_5$ or the symmetric group $\mathfrak{S}_4$ (see for example [36]). If $G_\lambda \neq G_\lambda^+$, then G_λ is isomorphic to either D_m, $C_m \times C_2$, $D_m \rtimes C_2$, $\mathcal{A}_4 \times C_2$, $\mathcal{A}_5 \times C_2$, $\mathfrak{S}_4$ or $\mathfrak{S}_4 \times C_2$.

As already seen in the previous section, for each $\sigma \in \mathfrak{S}_{n+1}$, such that $\Theta_n(\sigma) \in \mathbb{G}_n$ fixes $\lambda \in \Omega_n$, there is a (unique) Möbius transformation $M_{\sigma,\lambda} \in G_\lambda^+$. In the other direction, each $M \in G_\lambda^+$ induces a permutation $\sigma_M \in \mathfrak{S}_{n+1}$ by the following rule:

$$\left(M(p_{\sigma_M^{-1}(1)}), \ldots, M(p_{\sigma_M^{-1}(n+1)})\right) = (p_1, \ldots, p_{n+1}),$$

that is, $M = M_{\sigma_M,\lambda}$, this is verified, since if $M \in G_\lambda^+$, for $i \in \{1, ..., n+1\}$ we have that $M(p_i) = p_j$ for any $j \in \{1, ..., n+1\}$, so M induces a permutation $\sigma_M \in \mathfrak{S}_{n+1}$ such that $\sigma_M(i) = j$, with this we have that $M(p_{\sigma_M^{-1}(j)}) = M(p_i) = p_j$ for $j \in \{1, ..., n+1\}$. The above provides of an injective homomorphism

$$\xi_\lambda : \left\{ \begin{array}{ccc} G_\lambda^+ & \rightarrow & \mathfrak{S}_{n+1} \\ M & \mapsto & \sigma_M \end{array} \right. ,$$

the injectivity of ξ_λ is given, because if $M_1, M_2 \in G_\lambda^+$ are such that $\xi_\lambda(M_1) = \xi_\lambda(M_2)$, this is, $\sigma_{M_1} = \sigma_{M_2}$, we have that to σ_{M_1} corresponds to (unique) Möbius transformation $M_{\sigma_{M_1},\lambda}$ such that $M_{\sigma_{M_1},\lambda} = M_1 = M_{\sigma_{M_2},\lambda} = M_2$, therefore $M_1 = M_2$. On the other hand, ξ_λ is a homomorphism since it is fulfilled that $\xi_\lambda(M_1 \circ M_2) = \xi_\lambda(M_1) \circ \xi_\lambda(M_2)$, this is, $\sigma_{M_1 \circ M_2} = \sigma_{M_1} \circ \sigma_{M_2} = \sigma_{M_2}\sigma_{M_1}$ (product of permutations on the left). Thus, we must verify that the permutation associated with $M_1 \circ M_2$ is equal to the product of the permutations associated with M_2 and M_1. Since $(\sigma_{M_2}\sigma_{M_1})^{-1} = \sigma_{M_1}^{-1}\sigma_{M_2}^{-1}$, according to the procedure described in section (II.3.1) we have that, given the tuple $(p_1, ..., p_{n+1})$, acts on this first the permutation $\sigma_{M_1}^{-1}$ and then the permutation $\sigma_{M_2}^{-1}$, thus there is a (unique) Möbius transformation, M, such that sends the points $p_{\sigma_{M_2}^{-1}(\sigma_{M_1}^{-1}(1))} = \infty$, $p_{\sigma_{M_2}^{-1}(\sigma_{M_1}^{-1}(2))} = 0$ and $p_{\sigma_{M_2}^{-1}(\sigma_{M_1}^{-1}(1))} = 1$, and since $M \in G_\lambda^+$ we have that,

$$(M(p_{\sigma_{M_2}^{-1}(\sigma_{M_1}^{-1}(1))}), ..., M(p_{\sigma_{M_2}^{-1}(\sigma_{M_1}^{-1}(n+1))})) = (p_1, \cdots, p_{n+1}).$$

Also, as $M_1 \circ M_2(p_{\sigma_{M_2}^{-1}(\sigma_{M_1}^{-1}(i))} = p_i)$, for $i \in \{1, ..., n+1\}$, and by the uniqueness of M we have that $M = M_1 \circ M_2$ which verifies that ξ_λ is a homomorphism (see the next figure).

$$\begin{tikzcd}
(p_1, ..., p_{n+1}) \arrow[d, mapsto, "\sigma_{M_1}^{-1}"] & & \\
(p_{\sigma_{M_1}^{-1}(1)}, ..., p_{\sigma_{M_1}^{-1}(n+1)}) \arrow[d, mapsto, "\sigma_{M_2}^{-1}"] & & \\
(p_{\sigma_{M_2}^{-1}(\sigma_{M_1}^{-1}(1))}, ..., p_{\sigma_{M_2}^{-1}(\sigma_{M_1}^{-1}(n+1))}) \arrow[d, mapsto, "M \in PSL_2(\mathbb{C})"] & & \\
(M(p_{\sigma_{M_2}^{-1}(\sigma_{M_1}^{-1}(1))}), ..., M(p_{\sigma_{M_2}^{-1}(\sigma_{M_1}^{-1}(n+1))})) & = & (p_1, ..., p_{n+1})
\end{tikzcd}$$

Figure III.1. Möbius transformation, M, associated to the permutation $\sigma_{M_2}\sigma_{M_1}$

So, after post-composing ξ_λ with the isomomorphism $\Theta_n : \mathfrak{S}_{n+1} \to \mathbb{G}_n$, defines an injective homomorphism

$$\Theta_n \circ \xi_\lambda : \left\{ \begin{array}{ccc} G_\lambda^+ & \to & \mathbb{G}_n \\ M & \mapsto & \Theta_n(\xi_\lambda(M)) = \Theta_n(\sigma_M) \end{array} \right. ,$$

whose image, $(\Theta_n \circ \xi_\lambda)(G_\lambda^+) \subset \mathbb{G}_n$, is the $\mathbb{G}_n$-stabilizer, $\mathrm{Stab}_{\mathbb{G}_n}(\lambda)$, of the point $\lambda = (\lambda_1, \ldots, \lambda_{n-2})$, this is $(\Theta_n \circ \xi_\lambda)(G_\lambda^+) = \mathrm{Stab}_{\mathbb{G}_n}(\lambda)$, thus,

$$\mathrm{Stab}_{\mathbb{G}_n}(\lambda) = \{M \in PSL_2(\mathbb{C}) \mid M(C_\lambda) = C_\lambda\}.$$

Remark 8. The above (where $n \geq 4$) permits to observe the following facts (see also [47]).

(1) Let $\sigma \in \mathfrak{S}_{n+1}$ be different from the identity permutation and $T = \Theta_n(\sigma) \in \mathbb{G}_n$. It follows that T has order $m \geq 2$ and it has fixed points in Ω_n if and only if σ is in the conjugacy class of one of the following permutations.

(1.a) $(1, 2, \ldots, m)(m+1, \ldots, 2m) \cdots (rm+1, \ldots, (r+1)m)$, where $n = (r+1)m - 1$, some $r \in \{0, 1, \ldots\}$.

(1.b) $(1,2,\dots,m)(m+1,\dots,2m)\cdots(rm+1,\dots,(r+1)m)(n+1)$, where $n=(r+1)m$, some $r\in\{0,1,\dots\}$.

(1.c) $(1,2,\dots,m)(m+1,\dots,2m)\cdots(rm+1,\dots,(r+1)m)(n)(n+1)$, where $n=(r+1)m+1$, some $r\in\{0,1,\dots\}$.

This is verified, since, for the collection C_λ to remain invariant under a finite Möbius transformation, the points marked on the Riemann sphere $\widehat{\mathbb{C}}$ are located appropriately according to the figures (1.a), (1.b) and (1.c) of III.2 respectively. In each case the Möbius transformation (rotation around the vertical axis) leaves the collection of points invariant, which are arranged in cycles of m points, in the first case (1.a) the rotation does not fix any points, in the second case (1.b) the rotation fixes only one point and in the third case fixes two points. In this way the permutations described in (1.a), (1.b) and (1.c) respectively are associated for each case. See the figure III.2.

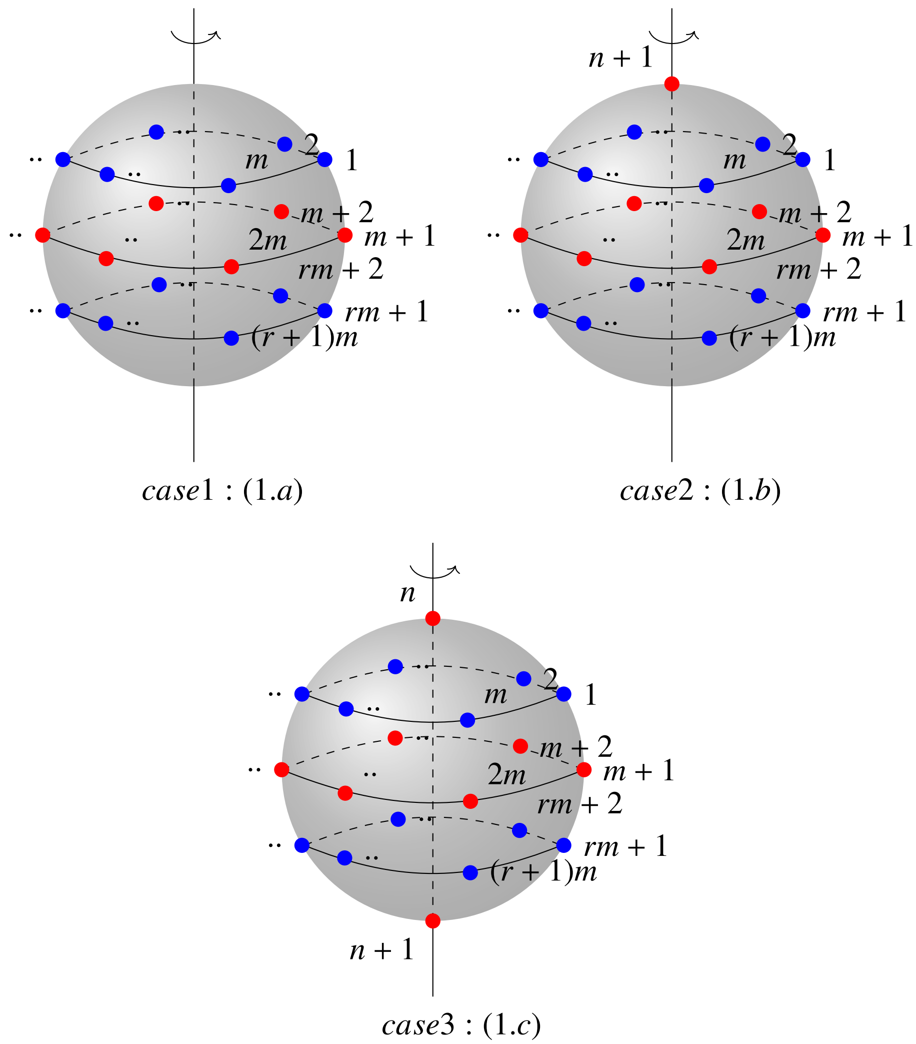


FIGURE III.2. Location of the $n+1$ marked points on the sphere whose rotation of order m induces the permutations of the classes (1.a) or (1.b) or (1.c) respectively

(2) About the $\mathbb{G}_n$-stabilizers, we have the following.

(2.a) If $n+1 \equiv \delta \mod (m)$, where $\delta \in \{0,1,2\}$, then we may find $\lambda \in \Omega_n$ with $\mathrm{Stab}_{\mathbb{G}_n}(\lambda) \cong C_m$.

This is a consequence of the above item, we have that the $n+1$ marked points remain invariant for a rotation of order m if $n+1 = k * m + \delta$, with $k \in \mathbb{N}$ and $\delta \in \{0,1,2\}$, where $\delta = 0$ means that the case (1.a) occurs, $\delta = 1$ corresponds to case (1.b) y $\delta = 2$ corresponds to case (1.c).

(2.b) If $n + 1 = 2mr + m\delta_1 + \delta_2$, where $\delta_1 \in \{0, 1, 2\}$ and $\delta_2 \in \{0, 2\}$, then we may find $\lambda \in \Omega_n$ with $\mathrm{Stab}_{\mathbb{G}_n}(\lambda) \cong D_m$.
(2.c) If $n + 1 = 12r + 6\delta_1 + 4\delta_2$, where $\delta_1 \in \{0, 1\}$ and $\delta_2 \in \{0, 1, 2\}$, then we may find $\lambda \in \Omega_n$ with $\mathrm{Stab}_{\mathbb{G}_n}(\lambda) \cong \mathcal{A}_4$.
(2.d) If $n + 1 = 24r + 12\delta_1 + 8\delta_2 + 6\delta_3$, where $\delta_1, \delta_2, \delta_3 \in \{0, 1\}$, then we may find $\lambda \in \Omega_n$ with $\mathrm{Stab}_{\mathbb{G}_n}(\lambda) \cong \mathfrak{S}_4$.
(2.e) If $n + 1 = 60r + 30\delta_1 + 20\delta_2 + 12\delta_3$, where $\delta_1, \delta_2, \delta_3 \in \{0, 1\}$, then we may find $\lambda \in \Omega_n$ with $\mathrm{Stab}_{\mathbb{G}_n}(\lambda) \cong \mathcal{A}_5$.

For the cases (2.c), (2.d) and (2.e) related to the groups $\mathcal{A}_4$, $\mathfrak{S}_4$ and $\mathcal{A}_5$ respectively, correspond to the rotations groups of a regular tetrahedron, octahedron or icosahedron inscribed in Riemann sphere $\widehat{\mathbb{C}}$ respectively. Thus, it is enough to verify how the $n + 1$ marked points in each of these polyhedra can be placed, so that the collection of points is invariant under the action of these groups. In addition, since the octahedron is conjugated from the cube and the icoshedron is conjugated from the dodecahedron. Below we will analyze the cases: (2.d) ($\mathfrak{S}_4$) and (2.c)($\mathcal{A}_4$).

The symmetries of the cube are: rotations around the axes that pass through the midpoints of opposite faces, of the axes passing through opposite vertices, and of the axes passing through midpoints of opposite edges. Therefore, we have 8 vertices that are in the same orbit, 6 medium points of faces that are in the same orbit, 12 average points of axes in the same orbit and finally we can place 4 points for each face that give a total of 24 points that are in the same orbit, like this: $n + 1 = 24r + 12\delta_1 + 8\delta_2 + 6\delta_3$, where $\delta_1, \delta_2, \delta_3 \in \{0, 1\}$ (see the figure III.3).

A regular tetrahedron has four axes of symmetry of order three, the straight lines perpendicular to each face by the opposite vertice of the tetrahedron, and 3 axes of symmetry of order two, these are the lines that join the midpoints of edges opposite. Therefore, we have 4 vertices that are in the same orbit, 4 medium points of faces that are in the same orbit, 6 average points of axes in the same orbit and finally we can place 3 points for each face that give a total of 12 points that are in the same orbit, like this: $n+1 = 12r+6\delta_1+4\delta_2$, where $\delta_1 \in \{0, 1\}$ and $\delta_2 \in \{0, 1, 2\}$ (see the figure III.4).

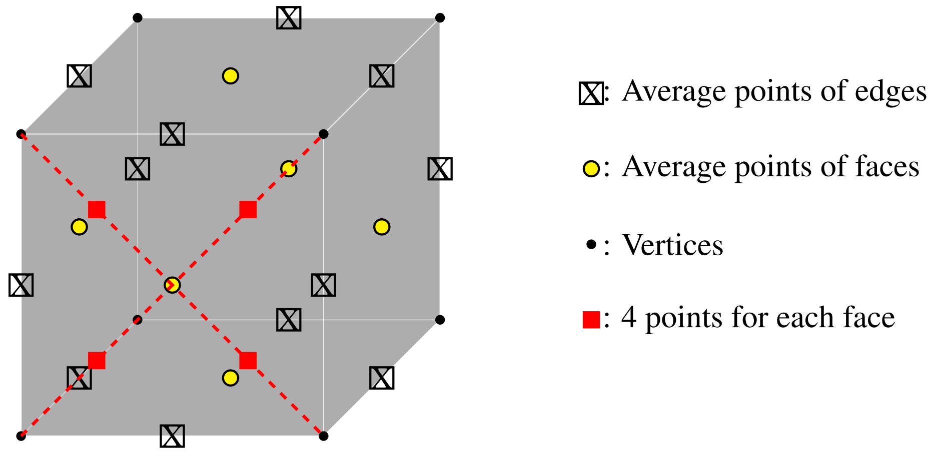


FIGURE III.3. Marked points invariant under the action of the symmetric group $\mathfrak{S}_4$.

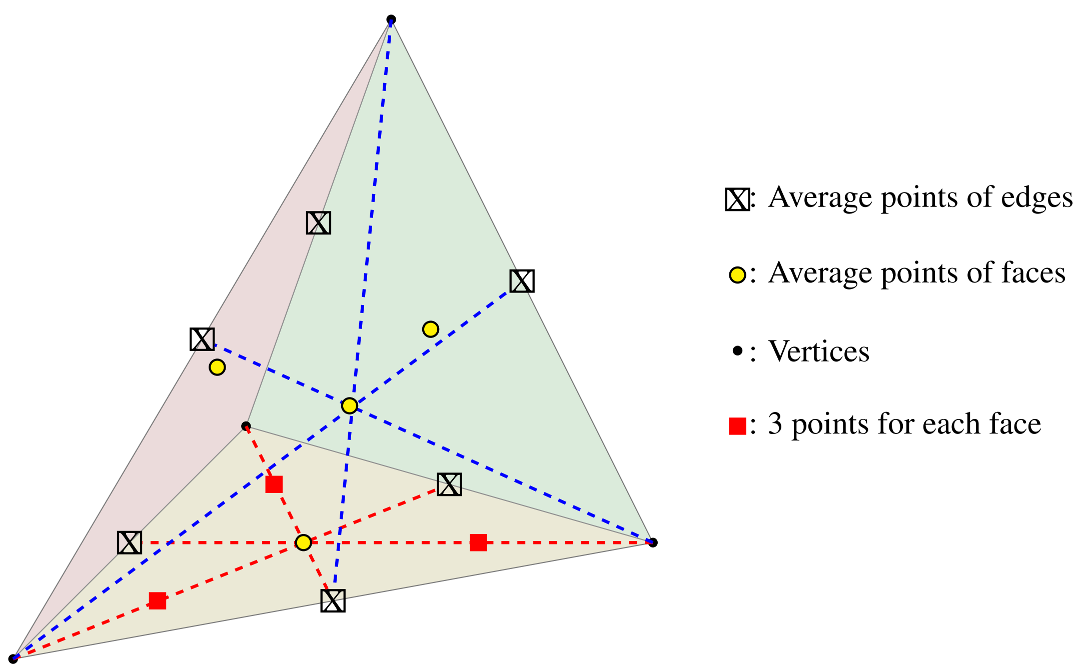


FIGURE III.4. Marked points invariant under the action of the alternating group $\mathcal{A}_4$.

Example 20. For $n = 4$, consider the order five automorphism $B = \Theta_4(\sigma)$, where $\sigma = (1,2,3,4,5)$. Then,

$$B(z_1, z_2) = \left(\frac{z_2}{z_2 - 1}, \frac{z_2}{z_2 - z_1}\right) \in \mathbb{G}_4,$$

$$\mathrm{Fix}(B) = \left\{\lambda := \left(\frac{1+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right),\ \mu := \left(\frac{1-\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}\right)\right\} \subset \Omega_4.$$

The order four element

$$S(z_1, z_2) = \left(\frac{1}{1 - z_2}, \frac{z_1 - 1}{z_2 - 1}\right) \in \mathbb{G}_4$$

satisfies that $S \circ B \circ S^{-1} = B^3$ and it permutes λ with μ. Each of these two points is stabilized by the dihedral group $\langle B, S^2 \rangle \cong D_5$.

As observed in the above example, for an element $T \in \mathbb{G}_n$ different from the identity and with fixed points in Ω_n, it might happen that its locus of fixed points is non-connected. But the two components (two points) are $\mathbb{G}_4$-equivalent. In [**47**] Schneps proved that the connected components of the locus of fixed points of T (each one a complex submanifold) forms an orbit under the action of the normalizing subgroup of $\langle T \rangle$ in $\mathbb{G}_n$ (for completeness, we provide a sketch of the proof since in [**47**] it is explicitly given only one of the cases).

III.1.1. The connectivity of the locus of fixed points.

Theorem 5. *For $n \geq 4$, let $\Theta_n(\sigma) = T \in \mathbb{G}_n$, of order $m \geq 2$ and $\mathrm{Fix}(T) \neq \emptyset$. Let $r + 1$, where $r \geq 0$, be the number of cycles of length m in the decomposition of σ (as in Remark 8). Then the following hold.*

(1) Each connected component of $\mathrm{Fix}(T)$ is a complex submanifold of Ω_n of dimension r.
(2) If $m = 2$, then $\mathrm{Fix}(T)$ is connected.
(3) If $m \geq 3$, then $\mathrm{Fix}(T)$ has exactly: (i) $\varphi(m)/2$ connected components if m divides $n + 1$, and (ii) $\varphi(m)$ connected components otherwise. Moreover, if F_1 and F_2 are any two of the connected components, then there is an element $S \in \mathbb{G}_n$, normalizing $\langle T \rangle$, such that $S(F_1) = F_2$.

Proof. Let $\sigma \in \mathfrak{S}_{n+1}$ be such that $T = \Theta_n(\sigma)$. Up to conjugation, we may assume that σ has one of the forms (see (1) of Remark 8)

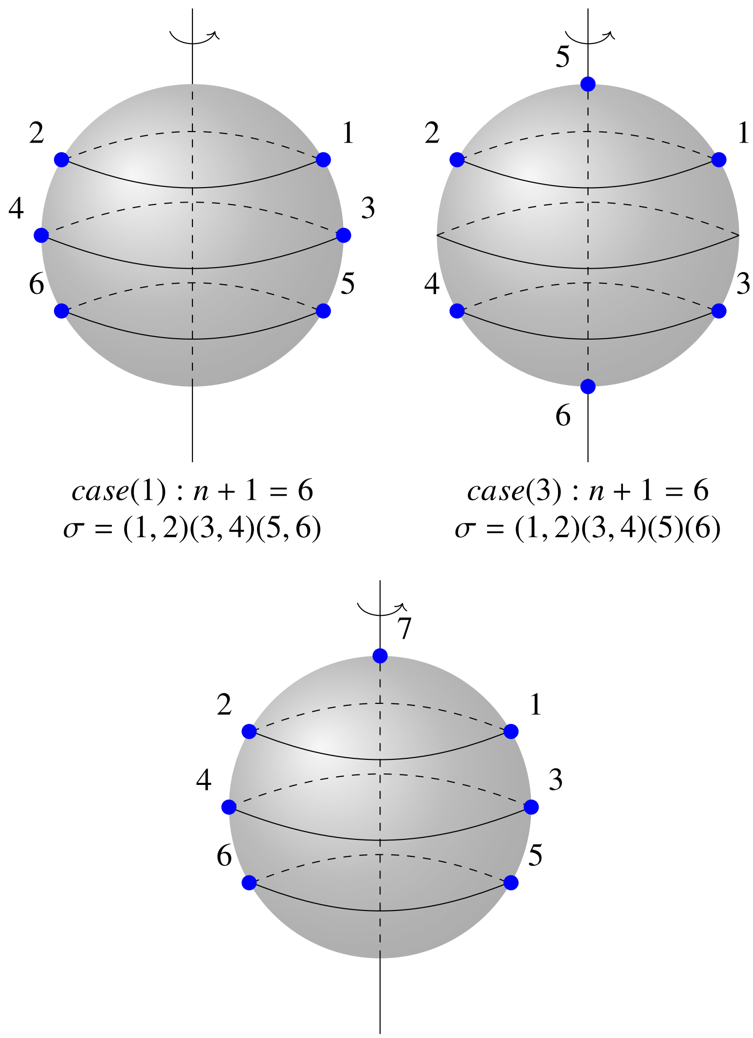
(1) $\sigma = (1, 2, \ldots, m) \cdots (rm + 1, \ldots, (r + 1)m)$, if $n = (r + 1)m - 1$.
(2) $\sigma = (1, 2, \ldots, m) \cdots (rm + 1, \ldots, (r + 1)m)(n + 1)$, if $n = (r + 1)m$.
(3) $\sigma = (1, 2, \ldots, m) \cdots (rm + 1, \ldots, (r + 1)m)(n)(n + 1)$, if $n = (r + 1)m + 1$.

Note that, for $m \geq 3$, only one of these possibilities may happen. For $m = 2$, both cases (1) and (3) happen for n odd and case (2) only happens for n even (see the figure III.5).

The image of a point $\lambda = (\lambda_1, \ldots, \lambda_{n-2}) \in \Omega_n$ under T is given by

$$T(\lambda_1, \ldots, \lambda_{n-2}) = \left(M_{\sigma,\lambda}(p_{\sigma^{-1}(4)}), \ldots, M_{\sigma,\lambda}(p_{\sigma^{-1}(n+1)})\right),$$

where $M_{\sigma,\lambda}$ is the (unique) Möbius transformation with $M_{\sigma,\lambda}(p_{\sigma^{-1}(1)}) = \infty$, $M_{\sigma,\lambda}(p_{\sigma^{-1}(2)}) = 0$, $M_{\sigma,\lambda}(p_{\sigma^{-1}(3)}) = 1$ and $p_1 = \infty$, $p_2 = 0$, $p_3 = 1$, $p_4 = \lambda_1, \ldots, p_{n+1} = \lambda_{n-2}$. Moreover, $\xi_\lambda(M_{\sigma,\lambda}) = \sigma$. In this way, $\mathrm{Fix}(T)$ consists of the tuples $(\lambda_1, \ldots, \lambda_{n-2}) \in \Omega_n$ such that the set $C_\lambda = \{\infty, 0, 1, \lambda_1, \ldots, \lambda_{n-2}\}$ is kept invariant under $M_{\sigma,\lambda}$.



$case(1) : n + 1 = 6$
$\sigma = (1,2)(3,4)(5,6)$

$case(3) : n + 1 = 6$
$\sigma = (1,2)(3,4)(5)(6)$

$case(2) : n + 1 = 7$
$\sigma = (1,2)(3,4)(5,6)(7)$

FIGURE III.5. For $m = 2$, cases (1),(2) and (3).

Case $m = 2$. Let us consider a point $\lambda \in \mathrm{Fix}(T)$. In this case, $M_{\sigma,\lambda}(x) = \lambda_1/x$, whose set of fixed points is $\mathrm{Fix}(M_{\sigma,\lambda}) = \left\{\pm\sqrt{\lambda_1}\right\}$.

Case (1), that is, $n-1 = 2r$, where $r \geq 2$. We have that $\sigma = (1,2)(3,4)...(2r+1,2r+2) = \sigma^{-1}$, thus, given $(z_1,\dots,z_{n-2}) \in \Omega_n$, we follow the procedure to find $T \in \mathbb{G}_n$:

$$
\begin{array}{c}
(\infty, 0, 1, \lambda_1, \lambda_2, \lambda_3, \dots, \lambda_{n-3}, \lambda_{n-2}) \\
\big\downarrow{\scriptstyle \sigma^{-1}=(1,2)(3,4)...(2r+1,2r+2)} \\
(0, \infty, \lambda_1, 1, \lambda_3, \lambda_2, \dots, \lambda_{n-2}, \lambda_{n-3}) \\
\big\downarrow{\scriptstyle M_{\sigma,\lambda}=\lambda_1/x} \\
(\infty, 0, 1, \lambda_1, \lambda_1/\lambda_3, \lambda_1/\lambda_2, \dots, \lambda_1/\lambda_{n-2}, \lambda_1/\lambda_{n-3})
\end{array}
$$

$$\Rightarrow T(\lambda_1,\dots,\lambda_{n-2}) = (\lambda_1, \lambda_1/\lambda_3, \lambda_1/\lambda_2, \dots, \lambda_1/\lambda_{n-2}, \lambda_1/\lambda_{n-3}).$$

We must have $\lambda_{2j+1} = \lambda_1/\lambda_{2j}$, for $j = 1,\dots,(n-3)/2$. So, the locus $\mathrm{Fix}(T)$ is homeomorphic to Ω_{r+2} by identifying the tuple $(\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_{n-2}) \in \mathrm{Fix}(T)$ with the tuple $(\lambda_1, \lambda_3, \lambda_5, \dots, \lambda_{n-2}) \in \Omega_{r+2}$. Since for any pair of elements of $\mathrm{Fix}(T)$ we can find a path that connects them, so it is arc-connected and therefore connected.

Case (2), that is, $n-2 = 2r$, where $r \geq 1$. We must have $\lambda_{2j+1} = \lambda_1/\lambda_{2j}$, for $j = 1,\dots,(n-4)/2$ and $\lambda_{n-2} \in \left\{\pm\sqrt{\lambda_1}\right\}$. We can move continuously λ_1 around the origin to pass from one of its square roots to the other. So, the locus $\mathrm{Fix}(T)$ is connected and provides a two fold cover of Ω_{r+2} by projecting the tuple $(\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_{n-2}) \in \mathrm{Fix}(T)$ to the tuple $(\lambda_1, \lambda_3, \lambda_5, \dots, \lambda_{n-3}) \in \Omega_{r+2}$.

Case (3), that is, $n-3 = 2r$, where $r \geq 1$. We must have $\lambda_{2j+1} = \lambda_1/\lambda_{2j}$, for $j = 1,\dots,(n-5)/2$ and $\lambda_{n-3}, \lambda_{n-2} \in \left\{\pm\sqrt{\lambda_1}\right\}$. Similarly as above, we may move continuously λ_1 around the origin to pass from one of its squre roots to the other. So, the locus $\mathrm{Fix}(T)$ is again connected and provides a two fold cover of Ω_{r+2} by projecting the tuple $(\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_{n-2}) \in \mathrm{Fix}(T)$ to the tuple $(\lambda_1, \lambda_2, \lambda_4, \dots, \lambda_{n-4}) \in \Omega_{r+2}$.

Case $m = 3$. Let us consider a point $\lambda \in \mathrm{Fix}(T)$. In this case, $M_{\sigma,\lambda}(x) = 1/(1-x)$, whose set of fixed points is $\mathrm{Fix}(M_{\sigma,\lambda}) = \left\{(1 \pm i\sqrt{3})/2\right\}$.

Case (1), that is, $n-2 = 3r$, where $r \geq 1$. We must have $\lambda_{3j-2} = 1/(1-\lambda_{3j})$ and $\lambda_{3j-1} = (\lambda_{3j}-1)/\lambda_{3j}$, for $j = 1,\dots,(n-2)/3$. So, the locus $\mathrm{Fix}(T)$, in this case, is homeomorphic to Ω_{r+2} by identifying the tuple $(\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_{n-2}) \in \mathrm{Fix}(T)$ with the tuple $(\lambda_3, \lambda_6, \lambda_9, \dots, \lambda_{n-2}) \in \Omega_{r+2}$.

Case (2), that is, $n-3 = 3r$, where $r \geq 1$. We must have we must have $\lambda_{3j-2} = 1/(1-\lambda_{3j})$ and $\lambda_{3j-1} = (\lambda_{3j}-1)/\lambda_{3j}$, for $j = 1,\dots,(n-3)/3$ and $\lambda_{n-2} \in \left\{(1 \pm i\sqrt{3})/2\right\}$. We can identify a tuple $(\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_{n-2}) \in \mathrm{Fix}(T)$ with the tuple $(\lambda_3, \lambda_6, \lambda_9, \dots, \lambda_{n-3}, \lambda_{n-2}) \in$

$\Omega_{r+2} \times \left\{(1 \pm i\sqrt{3})/2\right\}$. This provides two connected components, each one homeomorphic with Ω_{r+2}, these being permuted by the generator A in Lemma 1.

Case (3), that is, $n-4=3r$, where $r \geq 0$. We must have $\lambda_{3j-2} = 1/(1-\lambda_{3j})$ and $\lambda_{3j-1} = (\lambda_{3j}-1)/\lambda_{3j}$, for $j=1,\ldots,(n-4)/3$ and $\lambda_{n-3}, \lambda_{n-2} \in \left\{(1 \pm i\sqrt{3})/2\right\}$. We can identify a tuple $(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{n-2}) \in \mathrm{Fix}(T)$ with the tuple $(\lambda_3, \lambda_6, \ldots, \lambda_{n-4}, \lambda_{n-3}, \lambda_{n-2}) \in \Omega_{r+2} \times \left\{\left((1+i\sqrt{3})/2, (1-i\sqrt{3})/2\right), \left((1-i\sqrt{3})/2, (1+i\sqrt{3})/2\right)\right\}$ (where Ω_2 is just a singleton). This agains provides two connected components, each one homeomorphic with Ω_{r+2} which are permuted by the generator A in Lemma 1.

Case $m \geq 4$. Let us consider a point $\lambda \in \mathrm{Fix}(T)$. In this case, $M_{\sigma,\lambda}(x) = \lambda_{m-3}/(\lambda_{m-3}-x)$ and its set of fixed points is

$$\mathrm{Fix}(M_{\sigma,\lambda}) = \left\{p^{+}_{\sigma,\lambda} = \frac{\lambda_{m-3} + \sqrt{\lambda_{m-3}(\lambda_{m-3}-4)}}{2}, p^{-}_{\sigma,\lambda} = \frac{\lambda_{m-3} - \sqrt{\lambda_{m-3}(\lambda_{m-3}-4)}}{2}\right\}.$$

As $M_{\sigma,\lambda}$, of order $m \geq 3$, must preserve the set $\{\infty, 0, 1, \lambda_1, \ldots, \lambda_{m-3}\}$, there is an $M_{\sigma,\lambda}$-invariant circle Σ containing these points. As $\infty, 0, 1 \in \Sigma$, it follows that $\Sigma = \mathbb{R} \cup \{\infty\}$ and also that $M_{\sigma,\lambda}$ leaves invariant the upper half-plane $\mathbb{H}$. Let $p_{\sigma,\lambda} \in \mathrm{Fix}(M_{\sigma,\lambda})$ be the fixed point belonging to the upper half-plane $\mathbb{H}$.

Let C_0 (respectively, C_1) be the arc of circle starting at $p_{\sigma,\lambda}$ and ending at 0 (respectively, ending at 1) which is orthogonal to the real line. The angle between these two circles at $p_{\sigma,\lambda}$ is $2\alpha_\lambda\pi/m$, for some $\alpha_\lambda \in \{1, 2, \ldots, m-1\}$ relatively prime with m. This value α_λ determines uniquely the value of $\lambda_{m-3} = \lambda_{m-3}(\alpha_\lambda)$.

Let $\mathcal{L}_m$ be the set of points in $\{1, 2, \ldots, [(m-1)/2]\}$ relatively primes to m. As $M_{\sigma,\lambda}$ sends ∞ to 0 and 0 to 1, and it must preserve the orientation on the real line, it follows that $\alpha_\lambda \in \mathcal{L}_m$. Set $\mathrm{Fix}_{\alpha_\lambda}(T) \subset \mathrm{Fix}(T)$ the set of those $\widetilde{\lambda} \in \mathrm{Fix}(T)$ with $\alpha_{\widetilde{\lambda}} = \alpha_\lambda$ (so $\lambda \in \mathrm{Fix}_{\alpha_\lambda}(T)$).

Case (1), that is, $n+1 = m(r+1)$. If $r=0$, then the tuple $(\lambda_1, \ldots, \lambda_{n-2}) \in \mathrm{Fix}_{\alpha_\lambda}(T)$ is uniquely determined by α_λ. If $r=1$, then $(\lambda_1, \ldots, \lambda_{n-2}) \in \mathrm{Fix}_{\alpha_\lambda}(T)$ is uniquely determined by α_λ and $\lambda_{2m-3} \in \Omega_3$. If $r \geq 2$, then the tuple $(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{n-2}) \in \mathrm{Fix}_{\alpha_\lambda}(T)$ is uniquely determined by the tuple $(\lambda_{2m-3}, \lambda_{3m-3}, \ldots, \lambda_{(r+1)m-3} = \lambda_{n-2}) \in \Omega_{r+2}$. In this way, $\mathrm{Fix}_{\alpha_\lambda}(T)$ is homeomorphic to Ω_{r+2} (where Ω_2 is just a singleton), and the number of connected components of $\mathrm{Fix}(T)$ is the cardinality of $\mathcal{L}_m$, that is, $\varphi(m)/2$.

Case (2), that is, $n = m(r+1)$. If $r=0$, the tuple $(\lambda_1, \ldots, \lambda_{n-2}) \in \mathrm{Fix}_{\alpha_\lambda}(T)$ is uniquely determined by α_λ and the value of $\lambda_{n-2} \in \mathrm{Fix}(M_{\sigma,\lambda})$. If $r=1$, then $(\lambda_1, \ldots, \lambda_{n-2}) \in \mathrm{Fix}_{\alpha_\lambda}(T)$ is uniquely determined by α_λ, $\lambda_{2m-3} \in \Omega_3$, and $\lambda_{n-2} \in \mathrm{Fix}(M_{\sigma,\lambda})$. If $r \geq 2$, then the tuple $(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{n-2}) \in \mathrm{Fix}_{\alpha_\lambda}(T)$ is uniquely determined by the tuple $(\lambda_{2m-3}, \lambda_{3m-3}, \ldots, \lambda_{(r+1)m-3} = \lambda_{n-3}) \in \Omega_{r+2}$ and $\lambda_{n-2} \in \mathrm{Fix}(M_{\sigma,\lambda})$. In this way, we obtain that $\mathrm{Fix}_{\alpha_\lambda}(T)$ is homeomorphic to two disjoint copies of Ω_{r+2}. These two components are permuted by the element $\Theta_n(\tau)$, where $\tau \in \mathfrak{S}_{n+1}$ is such that $\tau^{-1}\sigma\tau = \sigma^{-1}$ (so, $\Theta_n(\tau) \circ T \circ \Theta_n(\tau)^{-1} = T^{-1}$). In this way, the number of connected components of $\mathrm{Fix}(T)$ is two times the cardinality of $\mathcal{L}_m$, that is, $\varphi(m)$.

Case (3), that is, $n-1 = m(r+1)$. If $r = 0$, the tuple $(\lambda_1, \ldots, \lambda_{n-2}) \in \mathrm{Fix}_{\alpha_\lambda}(T)$ is uniquely determined by α_λ, and the value of the pair $(\lambda_{n-3}, \lambda_{n-2}) \in \left\{(p^-_{\sigma,\lambda}, p^+_{\sigma,\lambda}), (p^+_{\sigma,\lambda}, p^-_{\sigma,\lambda})\right\}$. If $r = 1$, then the tuple $(\lambda_1, \ldots, \lambda_{n-2}) \in \mathrm{Fix}_{\alpha_\lambda}(T)$ is uniquely determined by α_λ, the value of $\lambda_{2m-3} \in \Omega_3$, and the valuer of the pair $(\lambda_{n-3}, \lambda_{n-2}) \in \left\{(p^-_{\sigma,\lambda}, p^+_{\sigma,\lambda}), (p^+_{\sigma,\lambda}, p^-_{\sigma,\lambda})\right\}$. If $r \geq 2$, then the tuple $(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{n-2}) \in \mathrm{Fix}_{\alpha_\lambda}(T)$ is uniquely determined by the tuple $(\lambda_{2m-3}, \lambda_{3m-3}, \ldots, \lambda_{(r+1)m-3}) \in \Omega_{r+2}$ and $(\lambda_{n-3}, \lambda_{n-2}) \in \left\{(p^-_{\sigma,\lambda}, p^+_{\sigma,\lambda}), (p^+_{\sigma,\lambda}, p^-_{\sigma,\lambda})\right\}$. We obtain that $\mathrm{Fix}_{\alpha_\lambda}(T)$ is homeomorphic to two disjoint copies of Ω_{r+2}. These two components are permuted by an element $\Theta(\tau)$ conjugating T to its inverse (as in the previous case). Again, the number of connected components of $\mathrm{Fix}(T)$ is two times the cardinality of $\mathcal{L}_m$, that is, $\varphi(m)$.

Let $\lambda, \mu \in \mathrm{Fix}(T)$ in different connected components. There are integers $\alpha_\lambda, \alpha_\mu \in \mathcal{L}_m$ such that $M_{\sigma,\lambda} = R_\lambda^{\alpha_\lambda}$ and $M_{\sigma,\mu} = R_\mu^{\alpha_\mu}$, where R_λ (respectively, R_μ) is the Möbius transformation of order m fixing the points $p_{\sigma,\lambda}$ and $\overline{p_{\sigma,\lambda}}$ (respectively, fixing the points $p_{\sigma,\mu}$ and $\overline{p_{\sigma,\mu}}$) which is rotation at angle $2\pi/m$ at $p_{\sigma,\lambda}$ (respectively, rotation at angle $2\pi/m$ at $p_{\sigma,\mu}$). It follows that there are integers $\beta_\lambda, \beta_\mu \in \{1, \ldots, m-1\}$, relatively primes to m, so that $R_\lambda = M_{\sigma,\lambda}^{\beta_\lambda}$ and $R_\mu = M_{\sigma,\mu}^{\beta_\mu}$ (in fact, $\alpha_\lambda\beta_\lambda \equiv 1 \ \mathrm{mod}(m)$ and $\alpha_\mu\beta_\mu \equiv 1 \ \mathrm{mod}(m)$). The image under $\Theta_n \circ \xi_\lambda$ of the transformation R_λ is T^{β_λ} and the image under $\Theta_n \circ \xi_\mu$ of R_μ is T^{β_μ}. As T^{β_λ} and T^{β_μ} both generates the cyclic group $\langle T \rangle$, there is an element $S \in \mathbb{G}_n$ such that $S \circ T^{\beta_\lambda} \circ S^{-1} = T^{\beta_\mu}$. It happens that S sends the set $\mathrm{Fix}_{\alpha_\lambda}(T)$ containing λ to the set $\mathrm{Fix}_{\alpha_\mu}(T)$ containing μ. □

Corollary 2. *Let $T = \Theta_n(\sigma) \in \mathbb{G}_n$, of order $m \geq 2$, with fixed points in Ω_n, where $n \geq 4$, and let $r \geq 0$ be such that in the decomposition of σ there are $(r+1)$ cycles of length m. Then the projection $\pi_n(\mathrm{Fix}(T)) = \mathcal{B}_{m,r} \subset \Omega_n/\mathbb{G}_n$ is a connected complex orbifold of dimension r.*

Next, we will see an example of Theorem 5 for $m \geq 4$.

Example 21 ($m \geq 4$ case (1))**.** Let $n = 6$, $m = 7$. Then $\sigma = (1, 2, 3, 4, 5, 6, 7)$ (σ does not fix any point), $T = \Theta_6(\sigma)$,

$$M_{\sigma,\lambda}(x) = \frac{\lambda_4}{\lambda_4 - x},$$

and

$$T(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \left(\frac{\lambda_4}{\lambda_4 - 1}, \frac{\lambda_4}{\lambda_4 - \lambda_1}, \frac{\lambda_4}{\lambda_4 - \lambda_2}, \frac{\lambda_4}{\lambda_4 - \lambda_3}\right).$$

As $\mathcal{L}_7 = \{1, 2, 3\}$, then $\mathrm{Fix}(T)$ has three connected components, that is,

$$\mathrm{Fix}(T) = \mathrm{Fix}_1(T) \cup \mathrm{Fix}_2(T) \cup \mathrm{Fix}_3(T),$$

where

$$\mathrm{Fix}_1(T) = \{\delta_1 := (\delta_{1,1}, \delta_{1,2}, \delta_{1,3}, \delta_{1,4})\},$$

$$\mathrm{Fix}_2(T) = \{\delta_2 := (\delta_{2,1}, \delta_{2,2}, \delta_{2,3}, \delta_{2,4})\}$$

and

$$\mathrm{Fix}_3(T) = \{\delta_3 := (\delta_{3,1}, \delta_{3,2}, \delta_{3,3}, \delta_{3,4})\},$$

where $\delta_{1,4}, \delta_{2,4}$ and $\delta_{3,4}$ correspond to the roots of the cubic equation $x^3 - 5x^2 + 6x - 1 = 0$ (these are x_1, x_2 and x_3 respectively), and for $j = 1, 2, 3$ we get $\delta_{j,1}, \delta_{j,2}$ and $\delta_{j,3}$ as:

$$\delta_{j,1} = \frac{\delta_{j,4}}{(\delta_{j,4} - 1)},$$

$$\delta_{j,2} = \frac{(\delta_{j,4} - 1)}{(\delta_{j,4} - 2)},$$

and

$$\delta_{j,3} = \frac{\delta_{j,4}}{(\delta_{j,4} - \delta_{j,2})},$$

So, we have that

$$M_{\sigma,\delta_1}(x) = \frac{\delta_{1,4}}{\delta_{1,4} - x},$$

$$M_{\sigma,\delta_2}(x) = \frac{\delta_{2,4}}{\delta_{2,4} - x},$$

and

$$M_{\sigma,\delta_3}(x) = \frac{\delta_{3,4}}{\delta_{3,4} - x}.$$

And the fixed points are:

$$\mathrm{Fix}(M_{\sigma,\delta_1}) = \{p^+_{\sigma,\delta_1}, p^-_{\sigma,\delta_1}\},$$

$$\mathrm{Fix}(M_{\sigma,\delta_2}) = \{p^+_{\sigma,\delta_2}, p^-_{\sigma,\delta_2}\}$$

$$\mathrm{Fix}(M_{\sigma,\delta_3}) = \{p^+_{\sigma,\delta_3}, p^-_{\sigma,\delta_1}\}.$$

Look the following figures that indicate the rotations of each M_{σ,δ_j} with respect to its fixed point p^+_{σ,δ_j}, where $j = 1, 2, 3$. In the three cases we have that each M_{σ,δ_j} leaves invariant of collection $C_{\delta_j} = \{\infty, 0, 1, \delta_{j,1}, \delta_{j,2}, \delta_{j,3}, \delta_{j,4}\}$ sending $\infty \mapsto 0 \mapsto 1 \mapsto \delta_{j,1} \mapsto \delta_{j,2} \mapsto \delta_{j,3} \mapsto \delta_{j,4} \mapsto \infty$.

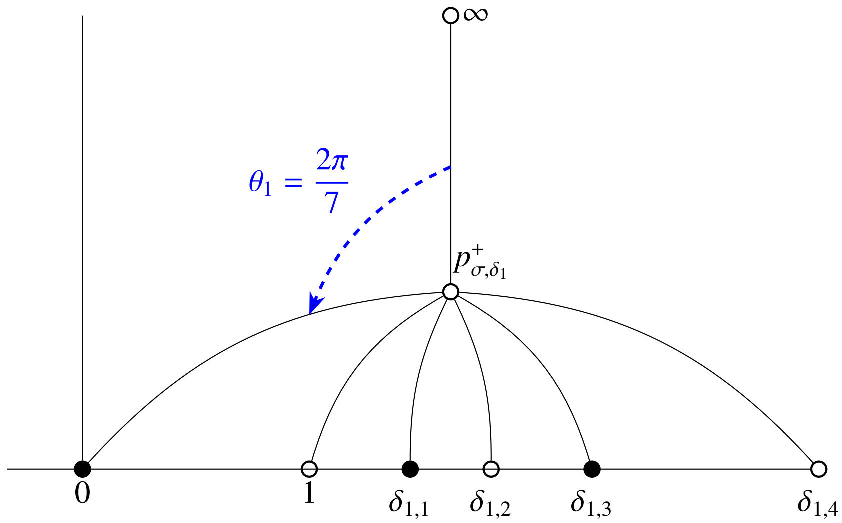


FIGURE III.6. Rotation M_{σ,δ_1} with angle of rotation $\theta_1 = \dfrac{2\pi}{7}$

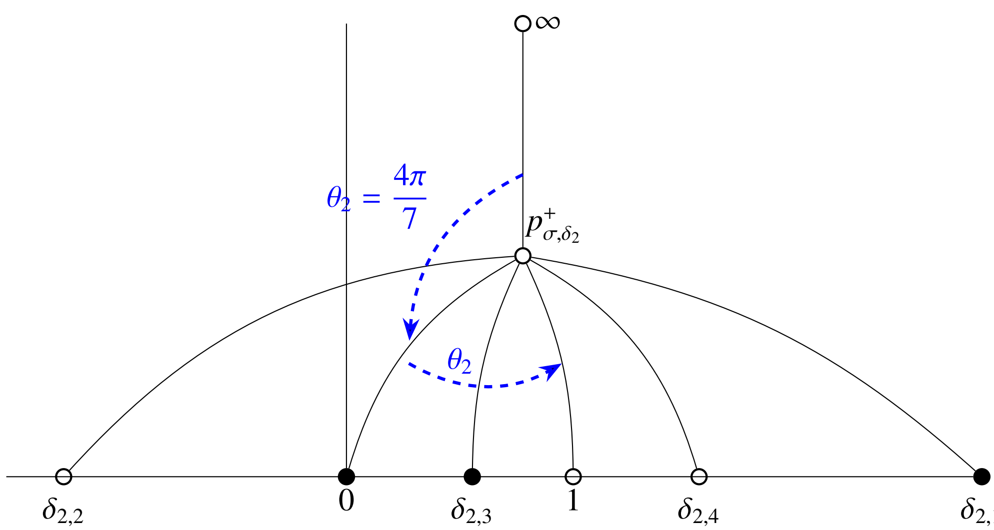


FIGURE III.7. Rotation M_{σ,δ_2} with angle of rotation $\theta_2 = \dfrac{4\pi}{7}$

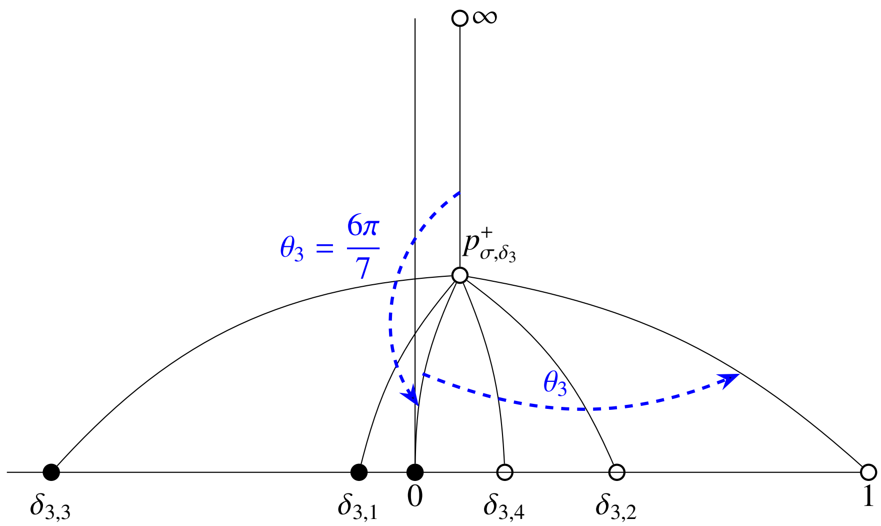


FIGURE III.8. Rotation M_{σ,δ_3} with angle of rotation $\theta_3 = \dfrac{6\pi}{7}$

As, $M_{\sigma,\delta_1} = R_{\delta_1}$, $M_{\sigma,\delta_2} = R^2_{\delta_2}$ and $M_{\sigma,\delta_3} = R^3_{\delta_3}$ with R_{δ_1}, R_{δ_2} and R_{δ_3} are rotations of order $m = 7$. Also we have that $M^1_{\sigma,\delta_1} = R_{\delta_1}$, $M^4_{\sigma,\delta_2} = R_{\delta_2}$ and $M^5_{\sigma,\delta_3} = R_{\delta_3}$, then, to R_{δ_1} corresponds to the application $T^1 \in \mathbb{G}_6$, to R_{δ_2} corresponds to the application $T^4 \in \mathbb{G}_6$ and to R_{δ_3} corresponds to the application $T^5 \in \mathbb{G}_6$.

The application $S \in \mathbb{G}_6$ such that permute the connected components $\mathrm{Fix}_1(T)$ and $\mathrm{Fix}_2(T)$, this is, $S(\mathrm{Fix}_1(T)) = (\mathrm{Fix}_2(T))$ satisfies that $S \circ T^1 S^{-1} = T^4$, so, we must find the permutation $\tau \in \mathfrak{S}_7$ such that $\tau^{-1}\sigma^1\tau = \sigma^4$. We have that $\tau = (1,7,3)(2,4,5)$, with this, to the permutation τ corresponds the application,

$$S(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = \left(\frac{\lambda_2(\lambda_4-1)}{(\lambda_4-\lambda_2)}, \frac{(\lambda_1-\lambda_2)(\lambda_4-1)}{(\lambda_1-1)(\lambda_4-\lambda_2)}, \frac{(\lambda_3-\lambda_2)(\lambda_4-1)}{(\lambda_3-1)(\lambda_4-\lambda_2)}, \frac{(\lambda_4-1)}{(\lambda_4-\lambda_2)}\right),$$

which sends δ_1 in δ_2.

Similarly, to find $Q \in \mathbb{G}_6$ such that permute the connected components $\mathrm{Fix}_2(T)$ and $\mathrm{Fix}_3(T)$, we should find the permutation $\gamma \in \mathfrak{S}_7$ such that $\gamma^{-1}\sigma^4\gamma = \sigma^5$. We have that $\gamma = (1,2,5,7,6,3)$, with this, to the permutation γ corresponds the application,

$$Q(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = \left(\frac{\lambda_3-1}{\lambda_1-1}, 1-\lambda_3, \frac{\lambda_3-1}{\lambda_4-1}, \frac{\lambda_3-1}{\lambda_2-1}\right),$$

which sends δ_2 in δ_3, therefore $\pi_n(Fix(T)) = B_{7,0}$ is connected.

Remark 9 (On Patterson's theorem)**.** Let us assume $n \geq 6$. Part (1) on the above theorem asserts that the locus of fixed points of a non-trivial element $\Theta_n(\sigma) \in \mathbb{G}_n$ has dimension r, where σ is a product of $(r+1)$ disjoint cycles, each one of length $m \geq 2$, with $(r+1)m \in \{n-1, n, n+1\}$. It can be checked that $n - 4 \geq r$, so the codimension of the locus of fixed points is at least two and, in particular, that $\mathcal{B}_{0,[n+1]}$ has codimension at least two. It follows from [44] that the singular locus of $\mathcal{M}_{0,[n+1]}$ coincides with the branch locus, obtaining Patterson's theorem [43, Theorem 3].

III.1.2. Main theorem connectivity of the Branch locus.

Theorem 6. *The branch locus $\mathcal{B}_{0,[n+1]}$ is connected if either (i) $n \geq 4$ is even or (ii) $n \geq 6$ is divisible by 3. It has exactly two connected components otherwise.*

Proof. **(A).** Let us denote by $\mathcal{B}_2$ the locus in $\Omega_n/\mathbb{G}_n$ obtained as the projection of those points being fixed by some involution. We proceed to see that it is a connected set. For $n \geq 4$ even, there is only one conjugacy class of involutions in $\mathbb{G}_n$ with fixed points, this corresponding to the permutation

$$\sigma = (1,2)(3,4)\cdots(n-1,n)(n+1).$$

So $\mathcal{B}_2 = \pi_n(\mathrm{Fix}(\Theta_n(\sigma))) = \mathcal{B}_{2,(n-2)/2}$, which is connected. Let us now assume $n \geq 5$ to be odd. In this case, there are two conjugacy classes of involutions in $\mathbb{G}_n$ with fixed points, these corresponding to the following two permutations in $\mathfrak{S}_{n+1}$:

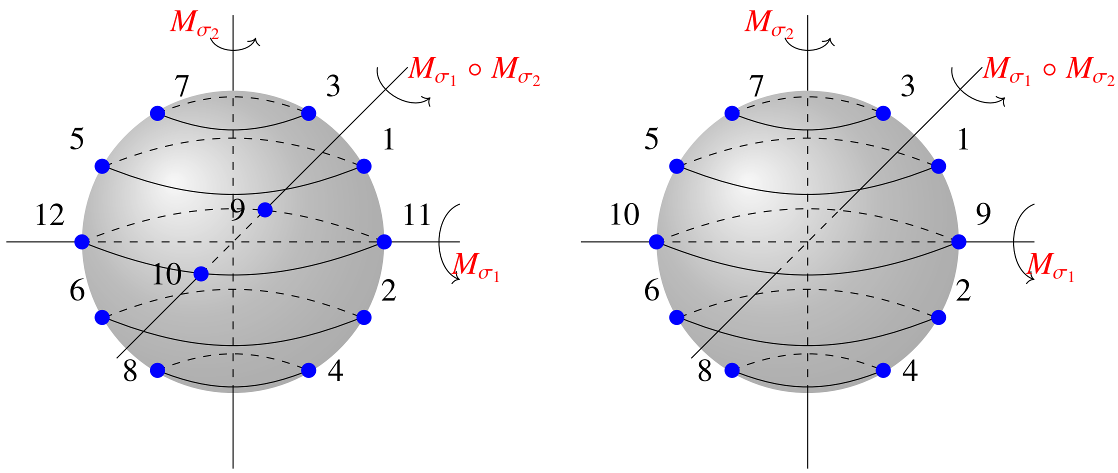
$$\sigma_1 = (1,2)(3,4)\cdots(n-2,n-1)(n)(n+1)$$

$$\sigma_2 = \begin{cases} (1,2s+1)(2,2s+2)\cdots(2s-1,4s-1)(2s,4s)(n,n+1), & n = 4s+1 \\ (1,2s+1)(2,2s+2)\cdots(2s-1,4s-1)(2s,4s)(n-2,n-1)(n,n+1), & n = 4s+3 \end{cases}$$

The involutions $\Theta_n(\sigma_1)$ and $\Theta_n(\sigma_2)$ induce, respectively, the connected sets $\mathcal{B}_{2,(n-3)/2}$ and $\mathcal{B}_{2,(n-1)/2}$ in $\Omega_n/\mathbb{G}_n$, so $\mathcal{B}_2 = \mathcal{B}_{2,(n-3)/2} \cup \mathcal{B}_{2,(n-1)/2}$. In order to get the connectivity of $\mathcal{B}_2$, we proceed to show that $\mathcal{B}_{2,(n-3)/2} \cap \mathcal{B}_{2,(n-1)/2} \neq \emptyset$.

As $\langle \sigma_1, \sigma_2 \rangle \cong C_2^2$, we have that $V_4 := \langle \Theta_n(\sigma_1), \Theta_n(\sigma_2) \rangle \cong C_2^2$. First, let us observe that $[\lambda = (\lambda_1, \ldots, \lambda_{n-2})] \in \mathcal{B}_{2,(n-3)/2} \cap \mathcal{B}_{2,(n-1)/2}$ if and only if the set $C_\lambda = \{\infty, 0, 1, \lambda_1, \ldots, \lambda_{n-2}\}$ is invariant under the Möbius trasnformations $M_1(x) = \lambda_1/x$ and $M_2(x) = (\lambda_{2s} - \lambda_{2s-2})(x - \lambda_{2s-1})/(\lambda_{2s} - \lambda_{2s-1})(x - \lambda_{2s-2})$ (and none of the points in the set C_λ is fixed by M_2). For instance, invariance under M_1 is guaranteed if $\lambda_{2j} = \lambda_1/\lambda_{2j+1}$, for $j = 1, \ldots, (n-5)/2$, $\lambda_{n-3} = \sqrt{\lambda_1}$ and $\lambda_{n-2} = -\sqrt{\lambda_1}$. In this way, we have freedom in the choices for the parameters $\lambda_1, \lambda_3, \lambda_5, \ldots, \lambda_{n-6}, \lambda_{n-4}$. Now, assuming the above conditions, M_2 has order two exactly if $\lambda_1 - \lambda_{2s-1}\lambda_{2s+1} - \lambda_{2s-1} + \lambda_{2s+1} = 0$. Under this extra assumption, we also have that $\langle M_1, M_2 \rangle \cong C_2^2$, $M_2(0) = \lambda_{2s-1}$ and $M_2(\lambda_1) = \lambda_{2s+1}$. If we set $\lambda_3 = M_2(\lambda_{2s+3}), \ldots, \lambda_{2s-3} = M_2(\lambda_{4s-3})$ and, in the case $n = 4s+3$, the points λ_{n-4} and λ_{n-5} are the fixed points of $M_2 \circ M_1$, that $\lambda_{2s-1} \pm \sqrt{\lambda_{2s-1}^2 - \lambda_1}$, then C_λ will be invariant under $\langle M_1, M_2 \rangle \cong C_2^2$ as desired.

In the figure III.9, for n is odd, the two possible cases for σ_1 and σ_2 are shown, as well as the distribution of points in the sphere. In this figure the rotations are observed M_{σ_1} and M_{σ_2} associated with σ_1 and σ_2 respectively and as the collection remains invariant under the action of the dihedral group $D_2 = C_2^2 \cong \langle M_{\sigma_1}, M_{\sigma_2} \rangle$.



$\sigma_1 = (1,2)(3,4)(5,6)(7,8)(9,10)(11)(12)$
$\sigma_2 = (1,5)(2,6)(3,7)(4,8)(9,10)(11,12)$
$n = 11(n = 4s+3)$

$\sigma_1 = (1,2)(3,4)(5,6)(7,8)(9)(10)$
$\sigma_2 = (1,5)(2,6)(3,7)(4,8)(9,10)$
$n = 9(n = 4s+1)$

FIGURE III.9. When n is odd, $\mathbb{Z}_2$ grows to a dihedral D_2.

(B). Let $T \in \mathbb{G}_n$ be of even order $2k$, where $k \geq 1$. If $\lambda \in \mathrm{Fix}(T)$, then λ is fixed under the involution T^k, in particular, $\pi_n(\mathrm{Fix}(T))$ intersects $\mathcal{B}_2$.

(C). If $T = \Theta_n(\sigma)$ has odd order $m \geq 3$, then we may assume, up to conjugation, that

(1) $\sigma = (1,2,\ldots,m)\cdots(rm+1,\ldots,(r+1)m)$, if $n = (r+1)m-1$.
(2) $\sigma = (1,2,\ldots,m)\cdots(rm+1,\ldots,(r+1)m)(n+1)$, if $n = (r+1)m$.
(3) $\sigma = (1,2,\ldots,m)\cdots(rm+1,\ldots,(r+1)m)(n)(n+1)$, if $n = (r+1)m+1$.

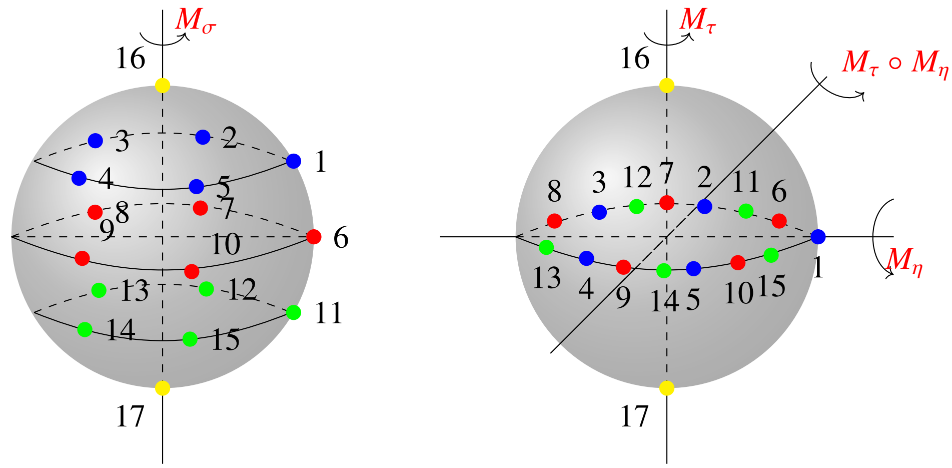
As before, $\pi_n(\mathrm{Fix}(T)) = \mathcal{B}_{m,r}$. Let $\tau \in \mathfrak{S}_{n+1}$ be the permutation, of order $(r+1)m$, defined as

$$\tau(lm+j) = (l+1)m+j, \quad j = 1,\ldots,m,\ l = 0,\ldots,r-1,$$
$$\tau(rm+j) = j+1,\ j = 1,\ldots,m-1,\ \tau((r+1)m) = 1.$$

In this way the permutation τ is the permutation of a cycle, whose matrix representation is as follows:

$$\tau = \begin{pmatrix} 1 & 2 & \ldots & m & m+1 & m+2 & \ldots & 2m & \ldots & rm+1 & rm+2 & \ldots & (r+1)m \\ m+1 & m+2 & \ldots & 2m & 2m+1 & 2m+2 & \ldots & 3m & \ldots & 2 & 3 & \ldots & 1 \end{pmatrix}$$

That is, the permutation τ sends each cycle of length m to the Ecuator circle (see the Figure: III.10 (b)).



(a) To the permutacion σ corresponds the rotation M_σ of order m.

(b) To the permutacion τ corresponds the rotation M_τ of order $(r+1)m$. And $\langle M_\tau, M_\eta \rangle \cong D_{(r+1)m}$.

FIGURE III.10. (C) The relation between the permutations σ and τ is $\sigma = \tau^{r+1}$.

Remark 10. (a) Note that $\Theta_n(\tau)$ has a non-empty locus of fixed points, contained inside the locus of fixed points of T, and: (i) for $n = (r+1)m - 1$, τ does not fixes any of the symbols, (ii) for $n = (r+1)m$, τ only fixes $n+1$ and (iii) for $n = (r+1)m + 1$, τ only fixes n and $n+1$. (b) It can be seen that $\sigma = \tau^{r+1}$, in particular, that $\mathcal{B}_{(r+1)m,0} \cap \mathcal{B}_{m,r} \neq \emptyset$. (c) If $n \in \{(r+1)m - 1, (r+1)m + 1\}$, then there is a permutation $\eta \in \mathfrak{S}_{n+1}$ of order two (of the same conjugacy class of either σ_1 or σ_2) such that $\langle \tau, \eta \rangle \cong D_{(r+1)m}$.

As a consequence of part (c) of Remark 10, if σ is as in cases (1) or (3), then $\mathcal{B}_{(r+1)m,0}$ intersects $\mathcal{B}_2$. Now, part (b) of the same remark asserts that $\mathcal{B}_{m,r} \cap \mathcal{B}_{(r+1)m,0} \neq \emptyset$. It follows that the sub-locus of $\mathcal{B}_{0,[n+1]}$, consisiting of the projections under π_n of the points being fixed by those automorphisms $\Theta_n(\sigma)$, where σ is either as in (1) or (3), is connected.

In order to obtain connectivity (or not) of $\mathcal{B}_{0,[n+1]}$, we need to study the locus of fixed points of those automorphisms coming from situation (2) above. So, let us assume $n = (r+1)m$ and σ as in (2).

The case $n \geq 4$ even. By part (b) of Remark 10, $\mathcal{B}_{m,r} \cap \mathcal{B}_{(r+1)m,0} \neq \emptyset$, and by (B) $\mathcal{B}_{(r+1)m,0} \cap \mathcal{B}_2 \neq \emptyset$. All the above then asserts that $\mathcal{B}_{0,[n+1]}$ is connected.

The case $n \geq 5$ odd. In this case, $r \geq 0$ is even and $m \geq 3$ odd. If $\mathcal{B}_{m,r} \cap \mathcal{B}_2 \neq \emptyset$, then there is a point $\lambda \in \mathrm{Fix}(T) \cap \mathrm{Fix}(S)$, where $S = \Theta_n(\rho)$, $\rho \in \mathfrak{S}_{n+1}$ is in the same conjugacy class of either σ_1 or σ_2 (so it has no fixed points or exactly two), and $\langle \sigma, \rho \rangle$ being isomorphic to either a cyclic group, a dihedral group, $\mathcal{A}_4$, $\mathcal{A}_5$ or $\mathfrak{S}_4$. The cyclic situation cannot happen as, in this case, ρ should also have only one fixed point, a contradiction. In the dihedral situation, ρ will have to permute two fixed points of σ, again a contradiction. In the cases $\mathfrak{S}_4$ and $\mathcal{A}_5$, there should be an involution in $\langle\sigma,\rho\rangle$ permuting two fixed points of σ, a contradiction. So the only possible situation is $\langle\sigma,\rho\rangle \cong \mathcal{A}_4$, $m = 3$ and $n = 3(1+2(s+2t))$, for a suitable $s \in \{0,1\}$ and $t \geq 0$, in which case, $\mathcal{B}_{3,2(s+2t)} \cap \mathcal{B}_2 \neq \emptyset$. As, by part (b) of Remark 10, $\mathcal{B}_{3,2(s+2t)} \cap \mathcal{B}_{n,0} \neq \emptyset$, we again obtain connectivity of $\mathcal{B}_{0,[n+1]}$ in the case n is divisible by 3.

In the complementary cases, that is, for $n \geq 5$ odd, relatively prime to 3, there is not a permutation in $\mathfrak{S}_{n+1}$ (in the conjugacy class of either σ_1 or σ_2) normalising $\langle\sigma\rangle$, in particular, $\mathcal{B}_{m,r} \cap \mathcal{B}_2 = \emptyset$, for all possibilities $n = m(r+1)$. As $\mathcal{B}_{n,0} \cap \mathcal{B}_{m,r} \neq \emptyset$, we obtain that $\mathcal{B}_{0,[n+1]}$ has exactly two connected components (one containing $\mathcal{B}_2$ and the other containing $\mathcal{B}_{n,0}$).

□

Next we will see two examples for n **odd**, in the first case we have that the branch locus has two connected components since n is not divisible by 3 and in the second example, n is divisible by 3, thus the branch locus is connected.

Example 22 (Theorem 6, n odd not divisible by 3)**.** Let $n = 11$, then $n+1 = 12$ and we have that the cyclic groups admissible as stabilizers of $\lambda \in \Omega_{11}$ are $C_2, C_3, C_4, C_5, C_6, C_{10}, C_{11}, C_{12} \subset \mathfrak{S}_{12}$. Now we will analyze cases (A), (B) and (C) of the proof of the previous theorem.

*(A)*For $m = 2$, in this case, there are two conjugacy classes of involutions in $\mathbb{G}_{11}$ corresponding to the permutations σ_1 and σ_2:

$$\sigma_1 = (1,2)(3,4)(5,6)(7,8)(9,10)$$

$$\sigma_2 = (1,5)(2,6)(3,7)(4,8)(9,10)(11,12)$$

with this $\mathcal{B}_2 = \mathcal{B}_{2,4} \cup \mathcal{B}_{2,5}$ is connected.

*(B)*If m is even, this is if $m \in \{4,6,10,12\}$,we have that each of the projections $\mathcal{B}_{4,2}, \mathcal{B}_{6,1}, \mathcal{B}_{10,0}, \mathcal{B}_{12,0}$ intersects with $\mathcal{B}_2$.

*(C)*If $T = \Theta_{11}(\sigma)$ has odd order $m \geq 3$, this is if $m \in \{3,5,11\}$, each associated permutation corresponds to case (1), (3) and (2) respectively,

$$\begin{array}{lll} m=3 & \sigma = (1,2,3)(4,5,6)(7,8,9)(10,11,12) & case(1), r=3 \\ m=5 & \sigma = (1,2,3,4,5)(6,7,8,9,10)(11)(12) & case(3), r=1 \\ m=11 & \sigma = (1,2,3,4,5,6,7,8,9,10,11)(12) & case(2), r=0 \end{array}$$

so, for $m = 3$ or $m = 5$, we have that $\mathcal{B}_{3,3} \cap \mathcal{B}_{12,0} \neq \emptyset$ and $\mathcal{B}_{5,1} \cap \mathcal{B}_{10,0} \neq \emptyset$. For $m = 11$, we have that $\mathcal{B}_{11,0} \cap \mathcal{B}_2 = \emptyset$. Therefore we have two connected components (see the next figure).

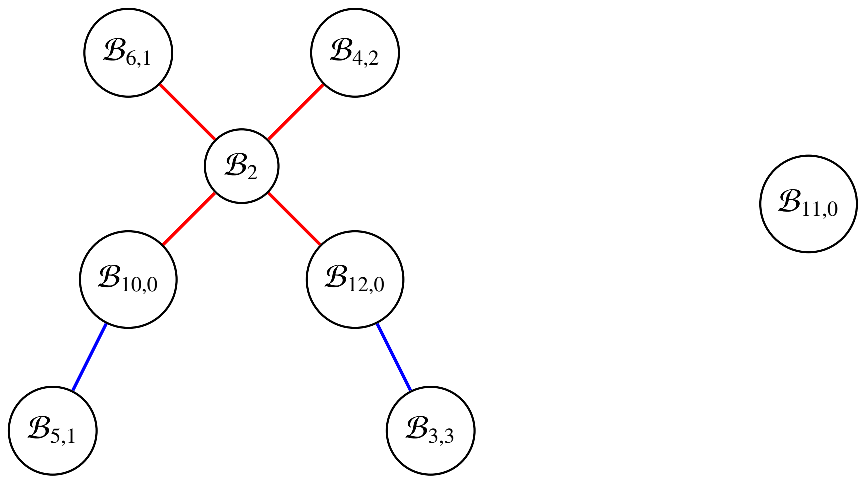


FIGURE III.11. Example conectivity of the Branch locus, n=11 , two connected components

Example 23 (Theorem 6, n odd divisible by 3)**.** Let $n = 27$, then $n + 1 = 28$ and we have that the cyclic groups admissible as stabilizers of $\lambda \in \Omega_{27}$ are $C_2, C_3, C_4, C_7, C_9, C_{13}, C_{14}, C_{26}, C_{27}, C_{28} \subset \mathfrak{S}_{28}$. Now we will analyze cases (A), (B) and (C) of the proof of the previous theorem.

*(A)*For $m = 2$, we have that $\mathcal{B}_2$ is connected.

*(B)*If m is even, this is if $m \in \{4, 14, 26, 28\}$,we have that each of the projections $\mathcal{B}_{4,6}, \mathcal{B}_{14,1}, \mathcal{B}_{26,0}, \mathcal{B}_{28,0}$ intersects with $\mathcal{B}_2$.

*(C)*If $T = \Theta_{27}(\sigma)$ has odd order $m \geq 3$, this is if $m \in \{3, 7, 9, 13, 27\}$, each associated permutation corresponds the following cases:

$$\begin{array}{ccc}
m = 3 & \sigma = (1, 2, 3)...(25, 26, 27)(28) & case(2), r = 8 \\
m = 7 & \sigma = (1, ..., 7)...(22, 23, 24, 25, 26, 27, 28) & case(1), r = 3 \\
m = 9 & \sigma = (1, ..., 9)...(19, 20, 21, 22, 23, 24, 25, 26, 27)(28) & case(2), r = 2 \\
m = 13 & \sigma = (1, ..., 13)(14, .., 26)(27)(28) & case(3), r = 1 \\
m = 27 & \sigma = (1, ..., 27)(28) & case(2), r = 0
\end{array}$$

so, for $m = 7$ and $m = 13$ (this is case (1) and (3)), we have that $\mathcal{B}_{7,3} \cap \mathcal{B}_{28,0} \neq \emptyset$ and $\mathcal{B}_{13,1} \cap \mathcal{B}_{26,0} \neq \emptyset$. For $m = 3$, we have that $\mathcal{B}_{3,8} \cap \mathcal{B}_2 \neq \emptyset$, also $\mathcal{B}_{3,8} \cap \mathcal{B}_{27,0} \neq \emptyset$ and $\mathcal{B}_{9,2} \cap \mathcal{B}_{27,0} \neq \emptyset$. Therefore we have the connectivity of the branch locus $\mathcal{B}_{0,[28]}$.

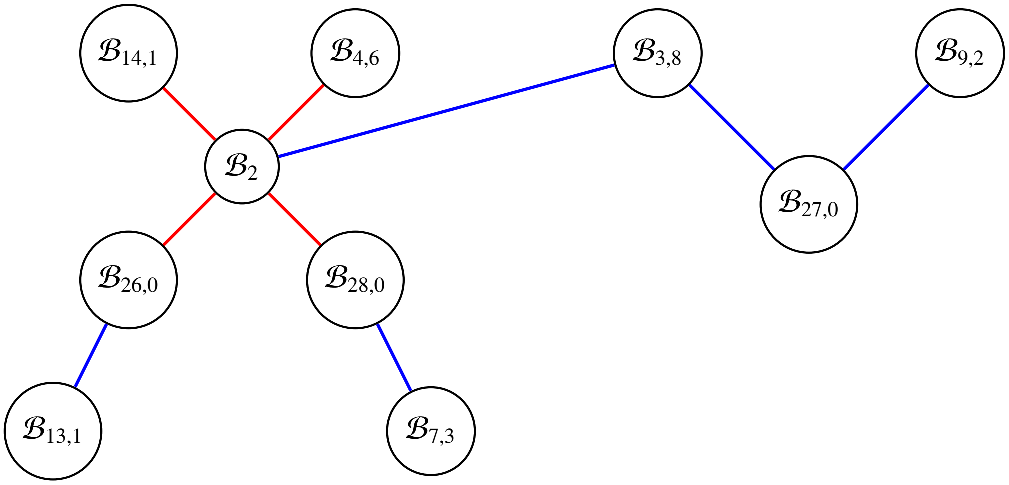


FIGURE III.12. Example conectivity of the Branch locus, n=27 , one connected component

III.2. The connectivity of the real locus

In this section we will present the results regarding the connectivity of the real locus, then we enunciate the main theorem:

Theorem 7. *If $n \geq 4$, then the following hold.*

(1) *The space Ω_n has exactly $[(n+3)/2]$ symmetries.*
(2) *The locus of fixed points of a symmetry of Ω_n is non-empty and each of its connected components is a real submanifold of real dimension $n-2$.*
(3) *If n is even, then the projection in $\mathcal{M}_{0,[n+1]}$ of the locus of fixed points of a symmetry of Ω_n is a connected real orbifold of dimension $n-2$. If n is odd, then the same holds for a symmetry, with the exception of those conjugated to*

$$S(z_1, \dots, z_{n-2}) = \left(\bar{z}_1, \frac{\bar{z}_1}{\bar{z}_3}, \frac{\bar{z}_1}{\bar{z}_2}, \frac{\bar{z}_1}{\bar{z}_5}, \frac{\bar{z}_1}{\bar{z}_4}, \dots, \frac{\bar{z}_1}{\bar{z}_{n-2}}, \frac{\bar{z}_1}{\bar{z}_{n-3}}\right),$$

for which the projection of its fixed points has two connected components, each one intersecting the projection of fixed points of the symmetry $J(z_1, \dots, z_{n-2}) = (\bar{z}_1, \dots, \bar{z}_{n-2})$.
(4) *The real locus $\mathcal{M}^{\mathbb{R}}_{0,[n+1]}$ is connected for $n \geq 5$ odd and it is not connected for $n = 2r$, $r \geq 5$ odd. If $p \geq 5$ is a prime, then $\mathcal{M}^{\mathbb{R}}_{0,[2p+1]}$ has exactly $(p-1)/2$ connected components.*

As a consequence of Remark 7, the above permits to obtain the following connectivity property of the locus of the real points of moduli space.

Corollary 3. *If $n = 2r$, where $r \geq 5$ is odd, then $\mathcal{M}_{0,[n+1]}(\mathbb{R}) = \mathcal{M}^{\mathbb{R}}_{0,[n+1]}$ is not connected. If $n \geq 5$ is odd, then $\mathcal{M}_{0,[n+1]}(\mathbb{R})$ is connected.*

In this section we proceed to prove Theorem 7.

III.2.1. Proof of Part (1) of Theorem 7. As previously noted, a symmetry of Ω_n has the form $T \circ J$, where $T = \Theta_n(\sigma) \in \mathbb{G}_n$ satisfies that $T^2 = I$ (that is, σ^2 is the identity permutation). As J commutes with every element of $\mathbb{G}_n$, two symmetries $S_1 = T_1 \circ J$ and $S_2 = T_2 \circ J$ are conjugated by elements of $\mathbb{G}_n$ if and only if the elements T_1 and T_2 are conjugated. It follows that the number of symmetries, up to conjugation by holomorphic automorphisms, is equal to one plus the number of conjugacy classes of elements of order two in the symmetric group $\mathfrak{S}_{n+1}$, that is, $[(n+3)/2]$ (this provides part (1) of Theorem 7).

III.2.2. Proof of Part (2) of Theorem 7. Up to conjugacy, we may assume

$$\sigma = (1,2)(3,4)\cdots(2\beta-1,2\beta)(2\beta+1)\cdots(n+1),\ \beta \in \{0,1,\ldots,[(n+1)/2]\},$$

where for $\beta = 0$ we mean σ the identity permutation. In this case,

(III.1)

$$T(\lambda_1,\ldots,\lambda_{n-2}) = \begin{cases} (\lambda_1,\ldots,\lambda_{n-2}), & \beta = 0. \\ \left(\frac{1}{\lambda_1},\frac{1}{\lambda_2},\ldots,\frac{1}{\lambda_{n-2}}\right), & \beta = 1. \\ \left(\lambda_1,\frac{\lambda_1}{\lambda_2},\frac{\lambda_1}{\lambda_3},\ldots,\frac{\lambda_1}{\lambda_{n-2}}\right), & \beta = 2. \\ \left(\lambda_1,\frac{\lambda_1}{\lambda_3},\frac{\lambda_1}{\lambda_2},\frac{\lambda_1}{\lambda_5},\frac{\lambda_1}{\lambda_4}\ldots,\frac{\lambda_1}{\lambda_{2s+1}},\frac{\lambda_1}{\lambda_{2s}},\frac{\lambda_1}{\lambda_{2s+2}},\frac{\lambda_1}{\lambda_{2s+3}},\frac{\lambda_1}{\lambda_{n-2}}\right), & s = \beta-2,\ \beta \geq 3. \end{cases}$$

and

$$\text{(III.2)}\quad S(\lambda_1,\ldots,\lambda_{n-2}) = \begin{cases} (\overline{\lambda}_1,\ldots,\overline{\lambda}_{n-2}), & \beta = 0. \\ \left(\frac{1}{\overline{\lambda}_1},\frac{1}{\overline{\lambda}_2},\ldots,\frac{1}{\overline{\lambda}_{n-2}}\right), & \beta = 1. \\ \left(\overline{\lambda}_1,\frac{\overline{\lambda}_1}{\overline{\lambda}_2},\frac{\overline{\lambda}_1}{\overline{\lambda}_3},\ldots,\frac{\overline{\lambda}_1}{\overline{\lambda}_{n-2}}\right), & \beta = 2. \\ \left(\overline{\lambda}_1,\frac{\overline{\lambda}_1}{\overline{\lambda}_3},\frac{\overline{\lambda}_1}{\overline{\lambda}_2},\ldots,\frac{\overline{\lambda}_1}{\overline{\lambda}_{2s+1}},\frac{\overline{\lambda}_1}{\overline{\lambda}_{2s}},\frac{\overline{\lambda}_1}{\overline{\lambda}_{2s+2}},\frac{\overline{\lambda}_1}{\overline{\lambda}_{2s+3}},\frac{\overline{\lambda}_1}{\overline{\lambda}_{n-2}}\right), & s = \beta-2,\ \beta \geq 3. \end{cases}$$

Let us denote by $\mathrm{Fix}(S) \subset \Omega_n$ the locus of fixed points of a symmetry S. The real locus $\mathcal{M}^{\mathbb{R}}_{0,[n+1]} \subset \Omega_n/\mathbb{G}_n$ is the union of all the π_n-images of these fixed sets. Set $\mathcal{F}_0 = \pi_n(\mathrm{Fix}(J))$.

Proposition 3. *If S is a symmetry of Ω_n, then $\mathrm{Fix}(S) \neq \emptyset$ and every connected component of $\mathrm{Fix}(S)$ is a real submanifold, of dimension $n-2$.*

Proof. Up to conjugation by a suitable element of $\mathbb{G}_n$, we may assume $S = T \circ J$, where T and S have the forms as in (III.1) and (III.2), respectively. In this way, $\lambda = (\lambda_1,\ldots,\lambda_{n-2}) \in \Omega_n$ is a fixed point of S if and only if $T(\lambda) = \overline{\lambda}$. Now, as $\mathrm{Fix}(J) = \Omega_n \cap \mathbb{R}^{n-2} \neq \emptyset$, we only need to take care of the case when T is different from the identity (so of order two). Let $\lambda = (\lambda_1,\ldots,\lambda_{n-2}) \in \Omega_n$. If $\beta = 1$, then $\lambda \in \mathrm{Fix}(S)$ if and only if $|\lambda_j| = 1$, $j = 1,\ldots n-2$. If $\beta = 2$, then $\lambda \in \mathrm{Fix}(S)$ if and only if $\lambda_1 \in (0,+\infty) \setminus$

$\{1\}$, $|\lambda_j| = \sqrt{\lambda_1}$, $j = 2, \ldots, n-2$. If $\beta \geq 3$, then $\lambda \in \mathrm{Fix}(S)$ if and only if $\lambda_1 \in (0, +\infty) \setminus \{1\}$, $\overline{\lambda}_3 = \frac{\lambda_1}{\lambda_2}, \overline{\lambda}_5 = \frac{\lambda_1}{\lambda_4}, \ldots, \overline{\lambda}_{2s+1} = \frac{\lambda_1}{\lambda_{2s}}$, and $|\lambda_j| = \sqrt{\lambda_1}$, $j = 2s+2, \ldots, n-2$. As in any of the above situations, the equations on the coordinates have solution, so we are done (see also Remark 11). □

III.2.3. Proof of Part (3) of Theorem 7.

Remark 11 (Fixed points description)**.** The above proof also permits to obtain a description of the locus of fixed points of the symmetries of Ω_n. For each $\lambda = (\lambda_1, \ldots, \lambda_{n-2}) \in \Omega_n$ we set $C_\lambda = \{p_1 = \infty, p_2 = 0, p_3 = 1, p_4 = \lambda_1, \ldots, p_{n+1} = \lambda_{n-2}\}$. Let us consider a symmetry $S = \Theta_n(\sigma) \circ J$, where $\sigma \in \mathfrak{S}_{n+1}$ is either the identity or a permutation of order two. Then

(1) If σ is the identity, that is, $S = J$, then $\lambda \in \mathrm{Fix}(S)$ if and only if $C_\lambda \subset \mathbb{R} \cup \{\infty\}$, that is, C_λ is point-wise fixed by the usual complex conjugation map $x \mapsto \overline{x}$. In this case, the connected components of fixed points corresponds to all possible orderings that the collection $\{\lambda_1, \ldots, \lambda_{n-2}\}$ has in $\mathbb{R} - \{0, 1\}$. To be more precise, let $\mathcal{L}$ be the collection of triples (I_1, I_2, I_3), where $I_1 = (i_1, \ldots, i_a)$, $I_2 = (i_{a+1}, \ldots, i_{a+b})$, $I_3 = (i_{a+b+1}, \ldots, i_{n-2})$ and $\{i_1, \ldots, i_{n-2}\} = \{1, \ldots, n-2\}$ (we permit some of them to be empty tuples). For each tuple $(I_1, I_2, I_3) \in \mathcal{L}$ we let $L(I_1, I_2, I_3)$ be the set of points $(\lambda_1, \ldots, \lambda_{n-2}) \in \mathrm{Fix}(J) = \Omega_n \cap \mathbb{R}^{n-2}$ such that $\lambda_{i_1} < \cdots < \lambda_{i_a} < 0 < \lambda_{i_{a+1}} < \cdots < \lambda_{i_{a+b}} < 1 < \lambda_{i_{a+b+1}} < \cdots < \lambda_{i_{n-2}}$. We may observe that $\mathrm{Fix}(J)$ is the disjoint union of all the sets $L(I_1, I_2, I_3)$, where $(I_1, I_2, I_3) \in \mathcal{L}$. Observe that, for a given tuple $(I_1, I_2, I_3) \in \mathcal{L}$ as above, we may find an element $T = \Theta_n(\sigma) \in \mathbb{G}_n$ (where the permutation σ is chosen to keep fix each of the indices 1, 2 and 3) such that $T(L(I_1, I_2, I_3)) = L((1, \ldots, a), (a+1, \ldots, a+b), (a+b+1, \ldots, n-2)) := L$. Now, given a point $(\lambda_1, \ldots, \lambda_{n-2}) \in L$, we have the ordered collection

$$\lambda_1 < \cdots < \lambda_a < 0 < \lambda_{a+1} < \cdots < \lambda_{a+b} < 1 < \lambda_{a+b+1} < \cdots < \lambda_{n-2}.$$

We may find a Möbius transformation in $\mathrm{PSL}_2(\mathbb{R})$ sending λ_{n-4} to 0, λ_{n-3} to 1 and λ_{n-2} to ∞. Such a Möbius transformation induces an element $T \in \mathbb{G}_n$ that sends L to $L((1, 2, \ldots, n-2), \emptyset, \emptyset)$. This permits to observe that all the connected components of $\mathrm{Fix}(J)$ are $\mathbb{G}_n$-equivalent.

(2) If σ has order two, it is a product of $\beta \geq 1$ disjoint transpositions, $2\beta < n+1$, and fixes each of the points $\{j_1, \ldots, j_{n+1-2\beta}\} \subset \{1, \ldots, n+1\}$, then $\lambda \in \mathrm{Fix}(S)$ if and only if there is a reflection (that is, conjugated to $z \mapsto \overline{z}$) keeping invariant the set C_λ and fixing exactly the $n+1-2\beta$ points $p_{j_1}, \ldots p_{j_{n+1-2\beta}}$. In this case, the connected components of fixed points corresponds to all possible ordering that the collection $\{p_{j_1}, \ldots p_{j_{n+1-2\beta}}\}$ has in the circle of fixed points of the reflection.

In the following figure, we see an example for $n = 6$, the collection of points C_λ in the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is invariant under the reflection $z \mapsto \lambda_1/\overline{z}$ (conjugated to $z \mapsto 1/\overline{z}$ and $z \mapsto \overline{z}$), fixing the points in the circle of radius $\sqrt{\lambda_1}$ and permuting two to two the rest the points.

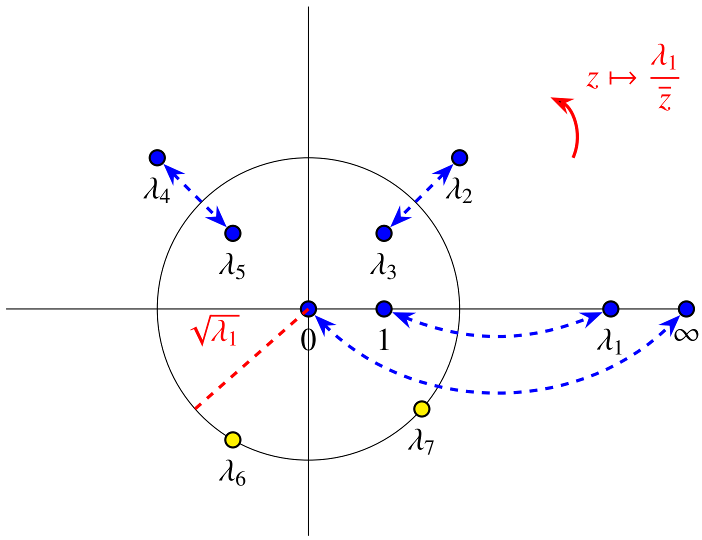


FIGURE III.13. Collection of points in $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ invariant under the reflection $z \mapsto \frac{\lambda_1}{\bar{z}}$, with $\lambda_1 \in (0, +\infty) \setminus \{1\}$

(3) If $n \geq 5$ is odd, $2\beta = n+1$, and σ is a product of β disjoint transpositions,this is, σ does not fix points, then $\lambda \in \mathrm{Fix}(S)$ if and only if there is either an imaginary reflection (that is, conjugated to $z \mapsto -1/\bar{z}$) or a reflection keeping invariant the set C_λ (and the reflection fixing none of them). By considering the model of S as in (III.2), we observe that $\mathrm{Fix}(S)$ has exactly three connected components:

$$A_1 := \left\{(\lambda_1, \ldots, \lambda_{n-2}) \in \Omega_n : \lambda_1 \in (-\infty, 0),\ \lambda_{2k+1} = \frac{\lambda_1}{\bar{\lambda}_{2k}}, k = 1, \ldots, (n-3)/2\right\}$$

$$A_2 := \left\{(\lambda_1, \ldots, \lambda_{n-2}) \in \Omega_n : \lambda_1 \in (0, 1),\ \lambda_{2k+1} = \frac{\lambda_1}{\bar{\lambda}_{2k}}, k = 1, \ldots, (n-3)/2\right\}$$

$$A_3 := \left\{(\lambda_1, \ldots, \lambda_{n-2}) \in \Omega_n : \lambda_1 \in (1, \infty),\ \lambda_{2k+1} = \frac{\lambda_1}{\bar{\lambda}_{2k}}, k = 1, \ldots, (n-3)/2\right\}$$

The component A_1 corresponds to the imaginary reflection case and the others two, A_2 and A_3, to the reflection one. The automorphism $L(\lambda_1, \ldots, \lambda_{n-2}) = (\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_{n-2}^{-1}) \in \mathbb{G}_n$ normalizes S and permutes A_2 with A_3. We may observe that inside each A_j there are points with all of its coordinates being real, in particular, $A_j \cap \mathrm{Fix}(J) \neq \emptyset$. It follows, from Proposition 3, that $\pi_n(\mathrm{Fix}(S))$ consists of exactly two real analytic submanifolds $\pi_n(A_1)$ and $\pi_n(A_2) = \pi_n(A_3)$, each one of dimension $n-2$, each one intersecting $\mathcal{F}_0$.

In the following figures (figures III.14 and III.15) we can see an example of elements of the connected components A_1, A_2 and A_3 for $n = 9$, seen as collections of points in the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, each collection remains invariant under the reflection $z \mapsto \lambda_1/\bar{z}$ with $\lambda_1 < 0$, $0 < \lambda_1 < 1$ and $\lambda_1 > 1$ respectively, furthermore it is observed how the application $L(z_1, ..., z_7) = (1/z_1, ..., 1/z_7) \in \mathbb{G}_9$ sends the element

$$\lambda = \left(\frac{1}{4}, \frac{1-i}{4}, \frac{1-i}{2}, -\frac{1+i}{4}, -\frac{1+i}{2}, \frac{-1+i}{5}, \frac{-5+5i}{8}\right)$$

of the connected component A_2 into another element

$$\tilde{\lambda} = \left(4, 2+2i, 1+i, -2+2i, -1+i, \frac{-5-5i}{2}, \frac{-4-4i}{5}\right)$$

of the connected component A_3 (see the figures III.15 (a) and (b)).

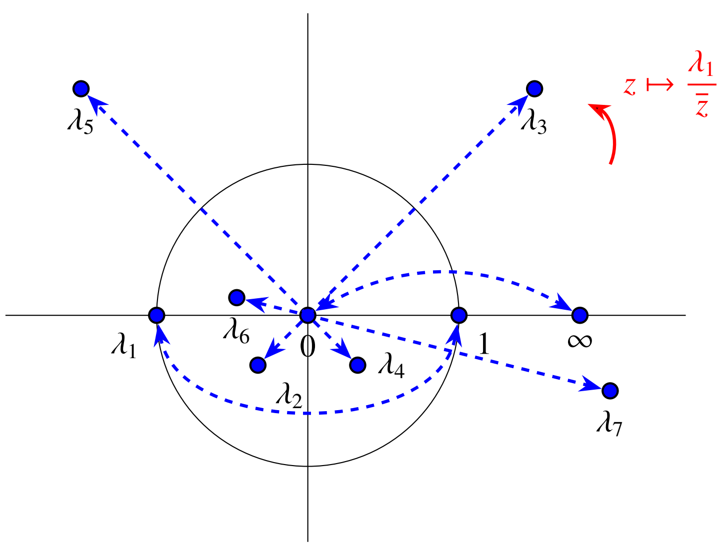
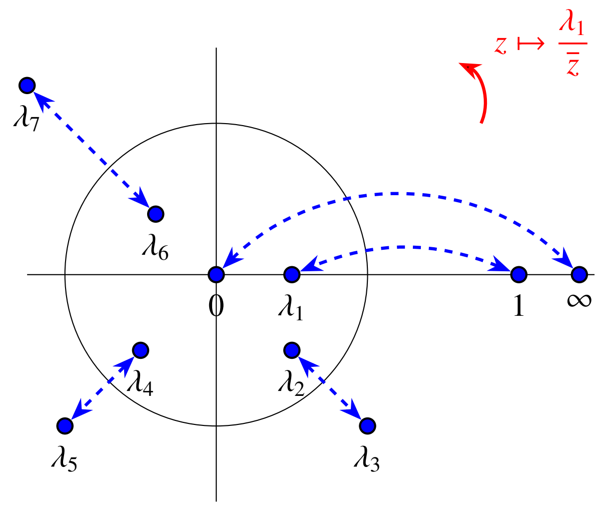
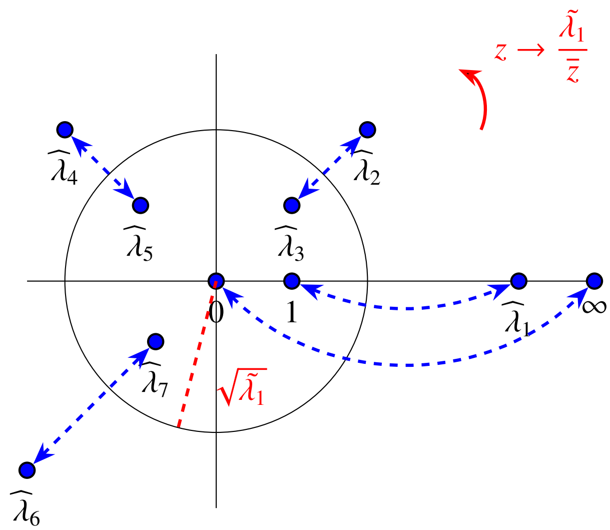


FIGURE III.14. Configuration of points of the connected component A_1 invariants under $z \mapsto \lambda_1/\bar{z}$ with $\lambda_1 < 0$



(a) A_2: Collection of points in $\widehat{\mathbb{C}}$ invariant under the reflection $z \mapsto \dfrac{\lambda_1}{\bar{z}}$, with $0 < \lambda_1 < 1$.



(b) A_3: Collection of points in $\widehat{\mathbb{C}}$ invariant under the reflection $z \mapsto \dfrac{\tilde{\lambda}_1}{\bar{z}}$, with $\tilde{\lambda}_1 > 1$.

FIGURE III.15. The application $L(z_1, ..., z_7) = (1/z_1, ..., 1/z_7)$ sends the component A_2 in A_3, so they are $\mathbb{G}_9-$equivalents. Note that $T(z) = 1/z$ conjugates $z \mapsto \lambda/z$ to $z \mapsto \lambda^{-1}/z$.

Proposition 4. *Let $S = \Theta_n(\sigma) \circ J$ be a symmetry of Ω_n, where $n \geq 4$, and let $\beta \in \{0, 1, \dots, [(n+1)/2]\}$ be such that σ is the product of β transpositions.*

(1) *If $2\beta \neq n+1$ and F_1 and F_2 are any two connected components of the locus of fixed points of S, then there is an element $L \in \mathbb{G}_n$, normalizing S, such that $L(F_1) = F_2$. In particular, the locus $\mathcal{F}_\beta := \pi_n(\mathrm{Fix}(S))$ is a connected real orbifold of dimension $n-2$.*

(2) *If $2\beta = n+1$, then $\mathrm{Fix}(S)$ consists of three connected components, A_1, A_2 and A_3 (as described in Remark 11). There is an element $L \in \mathbb{G}_n$, of order two and normalizing S, permuting the two components A_2 and A_3. There is no element of $\mathbb{G}_n$ that normalizes S and sending A_1 to any of the other two. Each A_j intersects $\mathrm{Fix}(J)$. In particular, $\pi_n(\mathrm{Fix}(S))$ consists of two connected real orbifolds of dimension $n-2$, say $\pi_n(A_1)$ and $\mathcal{F}_{(n+1)/2} := \pi_n(A_2) = \pi_n(A_3)$, each of them intersection $\mathcal{F}_0$. Moreover, if $n \geq 5$ is odd, then $\pi_n(A_1) \cap \mathcal{F}_\beta \neq \emptyset$.*

Proof. Up to conjugation, we may assume S to be as in (III.2). Part (1), for the case $\beta = 0$ (respectively, part (2)) was already observed in part (1) (respectively, part (3)) of Remark 11.

Let us prove part (1) for $\beta > 0$. In the case $\beta = 1$, we may see that the different connected components of $\mathrm{Fix}(S)$ correspond to the many different ways to display the values $\lambda_1, \dots, \lambda_{n-2}$ in the unit circle. But, by considering permutations of the form $\tau = (1)(2)(3)\widehat{\tau} \in \mathfrak{S}_{n+1}$, we may see that $\Theta_n(\tau)$ normalises the symmetry S and permutes these connected components. The situation is similar for cases $\beta = 2$ and $\beta \geq 3$. In the first case we need to use the permutations of the form $\tau = (1)(2)(3)(4)\widehat{\tau}$, $\tau = (1)(2)(3,4)\widehat{\tau} \in \mathfrak{S}_{n+1}$ and in the second one case we need to use permutations of the form $\tau = (1)(2)(3)(4)\tau_1\tau_2$, $\tau = (1)(2)(3,4)\tau_1\tau_2 \in \mathfrak{S}_{n+1}$, where τ_1 is the identity permutation on the set $\{5, \dots, 2\beta\}$ and τ_2 a permutation on the set $\{2\beta+1, \dots, n+1\}$.

Last part can be checked just by considering the Klein group $G = \langle U(z) = -1/\overline{z}, V(z) = 1/\overline{z}\rangle \cong C_2^2$. Then we only need to observe that it is possible to find a G-invariant collection of $n+1$ points with the property that $n+1-2\beta$ are fixed under the reflection V and the other 2β are permuted under it. So the result follows from the fixed point description in Remark 11. □

By Proposition 4 we observe the following. Let $S = \Theta_n(\sigma) \circ J$ be a symmetry of Ω_n and $\beta \in \{0, 1, \dots, [(n+1)/2]\}$ as above.

(1) If $2\beta \neq n+1$, then $\mathcal{F}_\beta = \pi_n(\mathrm{Fix}(S))$ is connected.
(2) If $n \geq 5$ is odd and $2\beta = n+1$, then $\mathcal{F}_{(n+1)/2} := \pi_n(A_2) = \pi_n(A_3)$ and $\pi_n(A_1)$ are both connected, they intersect and $\pi_n(\mathrm{Fix}(S)) = \mathcal{F}_{(n+1)/2} \cup \pi_n(A_1)$.
(3) If $n \geq 4$ is even, then the real locus $\mathcal{M}^{\mathbb{R}}_{0,[n+1]}$ is the union the $[(n+3)/2]$ connected real orbifolds $\mathcal{F}_\beta$, where $\beta \in \{0, 1, \dots, [(n+1)/2]\}$.
(4) If $n \geq 5$ is odd, then the real locus is the union the $(n+1)/2$ connected real orbifolds $\mathcal{F}_\beta$, where $\beta \in \{0, 1, \dots, (n+1)/2\}$, together the extra one $\pi_n(A_1)$. The

component $\mathcal{F}_0$ intersects both $\mathcal{F}_{(n+1)/2}$ and $\pi_n(A_1)$ and, moreover, $\pi_n(A_1)$ intersects all the other ones. In Figure III.16 we see a graph of connectivity of the irreducible components for this case.

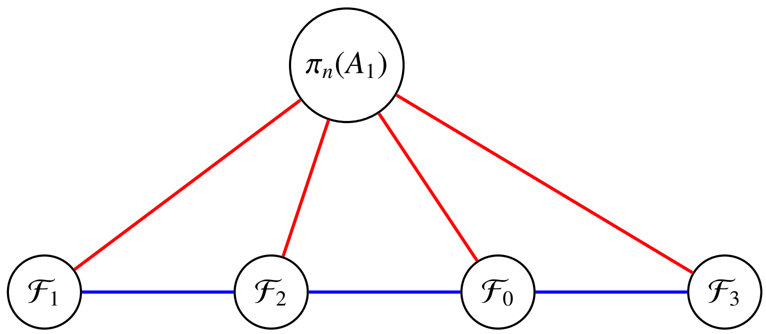


FIGURE III.16. Real locus connectivity graph for $n = 5$

The above asserts that in order to study the connectivity of the real locus, we only need to study the possible intersections between the components $\mathcal{F}_\beta$ (for n odd we must also consider the extra component $\pi_n(A_1)$). We call all these sets the "irreducible" components of $\mathcal{M}^{\mathbb{R}}_{0,[n+1]}$. The following result provides conditions for two of the irreducible components $\mathcal{F}_{\beta_1}$ and $\mathcal{F}_{\beta_2}$ to intersect.

Proposition 5. *Let $\beta_1, \beta_2 \in \{0, 1, \ldots, [(n+1)/2]\}$, $\beta_1 \neq \beta_2$. Then $\mathcal{F}_{\beta_1} \cap \mathcal{F}_{\beta_2} \neq \emptyset$ if and only if there are integers $m \geq 1$ and $\gamma \in \{0, 1, 2\}$ such that*

$$2m(\beta_1 + \beta_2) = (2m - 1)(n + 1 - \gamma). \tag{III.3}$$

PROOF. Let us start noting that $\mathcal{F}_{\beta_1} \cap \mathcal{F}_{\beta_2} \neq \emptyset$ is equivalent (see Remark 11) to have a point $\lambda \in \Omega_n$ such that the set C_λ is invariant under two reflections, τ_1 and τ_2, which are non-conjugated by a Möbius transformation keeping invariant the collection C_λ and

(i) τ_1 fixes pointwise $n + 1 - 2\beta_1$ of the points and permutes $2\beta_1$ of them;
(ii) τ_2 fixes pointwise $n + 1 - 2\beta_2$ of the points and permutes $2\beta_2$ of them.

The group $G = \langle \tau_1, \tau_2 \rangle$ is a subgroup of the stabilizer of C_λ, so it is a finite group; in fact a dihedral group of order $2r$, where r is the order of $\tau_2 \circ \tau_1$. As τ_1 and τ_2 are assumed to be non-conjugated, necessarily $r = 2m$, for some $m \geq 1$. In this way, there must be non-negative integers δ_1 and δ_2 and $\gamma \in \{0, 1, 2\}$, such that on the circle of fixed points of τ_1 there are $2\delta_1 + \gamma$ of the points of C_λ and on the circle of fixed points of τ_2 we must see $2\delta_2 + \gamma$ of points of that set, γ corresponds to if the fixed points of τ_1 and τ_2 intersect, that is, they could intersect in 1, 2 or no points, with respect to $2\delta_1$ and $2\delta_2$ are even numbers because they must be exchanged 2 to 2 with those of the other circle, that is (from first parts of (i) and (ii) above),

$$(*) \quad n+1-2\beta_1 = 2\delta_1+\gamma,\ n+1-2\beta_2 = 2\delta_2+\gamma,$$

and (from the second part of (i) and (ii)) that

$$(**) \quad 2\beta_1 = 2m\delta_2 + (2m-2)\delta_1,\ 2\beta_2 = 2m\delta_1 + (2m-2)\delta_2.$$

The equation $(**)$ is verified because, if we take the first circle of fixed points and transform it through a Möbius transformations to line L_1 and similarly take the other circle as the line L_2, they must intersect, let τ_1 and τ_2 be the reflections with respect to lines L_1 and L_2 respectively as in (i) and (ii) above, such that the order of $\tau_2 \circ \tau_1$ is $r = 2m$, we have that

$$2\beta_1 = (r\delta_1 - 2\delta_1) + r\delta_2 = 2m\delta_2 + (2m-2)\delta_1,$$

$$2\beta_2 = (r\delta_2 - 2\delta_2) + r\delta_1 = 2m\delta_1 + (2m-2)\delta_2,$$

where $(r\delta_1 - 2\delta_1)$ are the points that are on the lines $(\tau_2 \circ \tau_1)^k(L_1)$, for $k = 1, ..., (r-1)$ and $r\delta_2$ are the points that are on the lines $(\tau_2 \circ \tau_1)^k(L_2)$, for $k = 0, ..., (r-1)$, similarly for $2\beta_2$(see next example).

Equalities in $(*)$ impliy that

$$2\delta_1 = n+1-2\beta_1-\gamma,\ 2\delta_2 = n+1-2\beta_2-\gamma.$$

Plugging these in the equalities in $(**)$, we obtain the desired result. □

Below we see an example of the above proposition:

Example 24. Let $n = 7$, $\beta_1 = 2$ and $\beta_2 = 3$, then the collection C_λ of $n+1 = 8$ points is invariant for the reflections τ_1 and τ_2, where τ_1 fixed pointwise $n+1-2\beta_1 = 4$ of the points and permutes $2\beta_1 = 4$ of them, similarly τ_2 fixes pointwise $n+1-2\beta_2 = 2$ of the points and permutes $2\beta_2 = 6$ of them. Of $(*)$ we have that $\delta_1 = 1$ and $\delta_2 = 0$ with $\gamma = 2$ (γ fixed points in common) and by formula III.3 we have that $m = 3$. Let L_1 be the fixed point line of the reflection τ_1 related to β_1 and let L_2 be the fixed point line of the reflection τ_2 related to β_2, the order of the rotation $\tau_2 \circ \tau_1$ is $r = 2m = 6$, with angle of rotation $\theta = 2\pi/r = \pi/3$ (see the next figure).

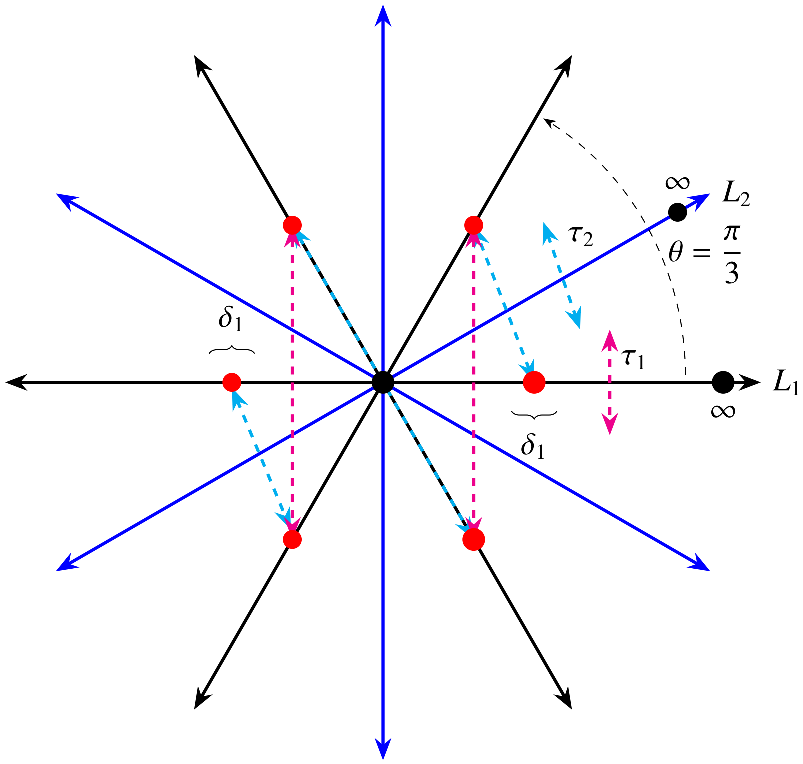


FIGURE III.17. Reflections τ_1 and τ_2 with respect to the lines L_1 and L_2

Remark 12. For equation (III.3) to have a solution, necessarily $n+1-\gamma$ must be divisible by $2m$, in particular: (i) for n even, we have $\gamma = 1$ and m a divisor of $n/2$, and (ii) for n odd, we have $\gamma \in \{0, 2\}$ and m a divisor of $(n+1-\gamma)/2$. So, for instance, (1) $\mathcal{F}_0 \cap \mathcal{F}_1 = \emptyset$, for $n \geq 4$, (2) $\mathcal{F}_1 \cap \mathcal{F}_2 \neq \emptyset$, if and only if $n \in \{4, 5, 6, 7\}$ and (3) $\mathcal{F}_0 \cap \mathcal{F}_\beta \neq \emptyset$, if and only if $\beta \in \{n-1, n, n+1\}$.

III.2.4. Proof of the connectivity of $\mathcal{M}^{\mathbb{R}}_{0,[n+1]}$ for $n \geq 5$ odd. If $\beta \in \{0, \ldots, (n-1)/2\}$, then $(n-1)/2 - \beta \in \{0, \ldots, (n-1)/2\}$ and , by using $m = 1$ and $\gamma = 2$ in (III.3), we obtain that $\mathcal{F}_\beta \cap \mathcal{F}_{(n-1)/2-\beta} \neq \emptyset$. Now, by using $m = 1$ and $\gamma = 0$, we obtain that $\mathcal{F}_{(n-1)/2-\beta} \cap \mathcal{F}_{\beta+1} \neq \emptyset$. In this way, we may connect using two edges the vertices $\mathcal{F}_\beta$ and $\mathcal{F}_{\beta+1}$, for $\beta \in \{0, \ldots, (n-1)/2\}$. Since the component $\pi_n(A_1)$ intersects $\mathcal{F}_0$ (in fact, it intersects all the other irreducible components), we obtain the connectivity of $\mathcal{M}^{\mathbb{R}}_{0,[n+1]}$.

III.2.5. $\mathcal{M}^{\mathbb{R}}_{0,[n+1]}$ is usually non-connected for $n \geq 4$ even. In the case $n \geq 4$ even, the connectivity of $\mathcal{M}^{\mathbb{R}}_{0,[n+1]}$ is described by the intersection graph $\mathcal{G}_n$ of $\mathcal{M}^{\mathbb{R}}_{0,[n+1]}$, whose set V_n of vertices are the values $\beta \in \{0, 1, \ldots, [(n+1)/2]\}$. Two different vertices $\beta_1, \beta_2 \in V_n$ are joined by an edge if the irreducible components $\mathcal{F}_{\beta_1}$ and $\mathcal{F}_{\beta_2}$ intersect. The intersection graph $\mathcal{G}_n$ describes how the different irreducible components intersect. Proposition 5 states necessary and sufficient conditions for two different irreducible components to intersect, in particular, it permits to describe the edges of the graph intersection $\mathcal{G}_n$. Some of these graphs are despicted in the Figures III.18, III.19 and III.20.

FIGURE III.18. Intersection graphs $\mathcal{G}_6$

FIGURE III.19. Intersection graphs $\mathcal{G}_{10}$

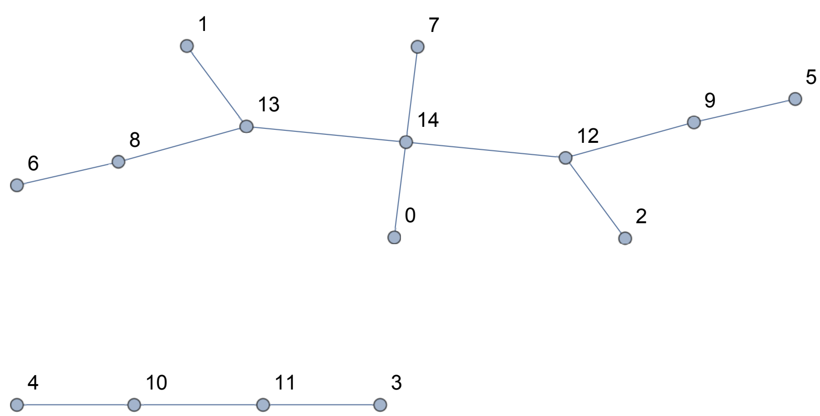


FIGURE III.20. Intersection graphs $\mathcal{G}_{28}$

Proposition 6 (Non-connectedness for $n = 2r \geq 4$, r odd)**.** *If $n = 2r$, where $r \geq 5$ is an odd integer, then $\mathcal{M}^{\mathbb{R}}_{0,[n+1]}$ is not connected. Moreover, for $r = p$, where p is a prime integer, the real locus $\mathcal{M}^{\mathbb{R}}_{0,[2p+1]}$ has exactly $(p-1)/2$ connected components.*

Proof. In this case $[(n+1)/2] = r$. By formula (III.3) and part (i) in Remark 12, for $\beta_1, \beta_2 \in \{0, 1, \ldots, r\}$, $\beta_1 \neq \beta_2$, the condition $\mathcal{F}_{\beta_1} \cap \mathcal{F}_{\beta_2} \neq \emptyset$ is equivalent to have $\beta_1 + \beta_2 = (2m-1)r/m$, where $m \geq 1$ is a divisor of r (so m must be odd). By taking $m = 1$, we obtain that $\mathcal{F}_{(r-1)/2} \cap \mathcal{F}_{(r+1)/2} \neq \emptyset$. We claim that none of these two can intersect other of the components. We check this for $(r-1)/2$ as for the other the argument is similar. Assume $\mathcal{F}_{(r-1)/2}$ intersects $\mathcal{F}_{\beta}$ for some $\beta \neq (r+1)/2$. Then, there must be a divisor $m \geq 1$ of r such that $(r-1)/2 + \beta = (2m-1)r/m$. It follows that $\beta = (m(3r+1) - 2r)/2m$, and as $\beta \leq r$, it follows that $m \leq 2r/(r+1) < 2$, a contradiction. If $r = p$, where $p \geq 3$ is a prime, then formula (III.3) reads as $m(\beta_1 + \beta_2) = (2m-1)p$, so $m \in \{1, p\}$. In this way, $\mathcal{F}_{\beta_1} \cap \mathcal{F}_{\beta_2} \neq \emptyset$ if and only if $\beta_1 + \beta_2 \in \{p, 2p-1\}$. Using $m = 1$, we obtain that $\mathcal{F}_{\beta} \cap \mathcal{F}_{p-\beta} \neq \emptyset$, for every $\beta \in \{0, \ldots, p\}$. By using $m = p$, we obtain that $\mathcal{F}_{\beta} \cap \mathcal{F}_{2p-1-\beta} \neq \emptyset$, for $\beta \in \{p-1, p\}$. It can be seen that $\{0, p, p-1, 1\}$ corresponds to one connected component of $\mathcal{M}^{\mathbb{R}}_{0,[2p+1]}$ and the others correspond to the sets $\{2, p-2\}, \{3, p-3\}, \ldots, \{(p-1)/2, (p+1)/2\}$. So, the number of connected components is exactly $(p-1)/2$. $\square$

All the above finishes the proof of Theorem 7.

Remark 13. As it was mentioned by one of the referees of the paper [2], where this part of the thesis is published, it is possible to state a more precise description of the connectivity of $\mathcal{M}^{\mathbb{R}}_{0,[n+1]}$, for $n = 2r$ and $r \geq 5$ odd in a similar way as in Theorem 5. We leave this task to the curious reader.

Proposition 7. *If $n = 4p$, where $p \geq 2$ is a prime integer, then $\mathcal{M}^{\mathbb{R}}_{0,[n+1]}$ is not connected if and only if $p \geq 7$.*

Proof. If $A, B \subset \{0, \ldots, 2p\}$, a map $E : A \to B$ is called a connectivity operator if for $\beta \in A$ we have that $\mathcal{F}_{\beta} \cap \mathcal{F}_{E(\beta)} \neq \emptyset$. Formula (III.3) asserts that $\mathcal{F}_{\beta_1} \cap \mathcal{F}_{\beta_2} \neq \emptyset$, for $\beta_1, \beta_2 \in \{0, \ldots, 2p\}, \beta_1 \neq \beta_2$, if and only if $\beta_1 + \beta_2 = (2m-1)2p/m$, where $m \in \{1, 2, p, 2p\}$. Each of the values of m induces a connectivity operator as follows

$$\begin{array}{lcl} E_1 & : & \beta \in \{0, 1, \ldots, 2p\} \mapsto 2p - \beta \in \{0, 1, \ldots, 2p\}, \\ E_2 & : & \beta \in \{p, p+1, \ldots, 2p\} \mapsto 3p - \beta \in \{p, p+1, \ldots, 2p\}, \\ E_p & : & \beta \in \{2p-2, 2p-1, 2p\} \mapsto 4p - 2 - \beta \in \{2p-2, 2p-1, 2p\}, \\ E_{2p} & : & \beta \in \{2p-1, 2p\} \mapsto 4p - 1 - \beta \in \{2p-1, 2p\}. \end{array}$$

Using the above connectivity operators, it can be checked that, for $k \in \{3, \ldots, p-3\}$ and $p \geq 7$, the vertices in $\{k, 2p-k, p+k, p-k\}$ defines a connected component. The connectivity for cases $p \in \{2, 3, 5\}$ can be checked directly by the connectivity operators. $\square$

Remark 14. In the case that $n = 4r$, where $r \geq 1$ is odd, but different from a prime, we may use Proposition 5 in order to observe that $\mathcal{M}^{\mathbb{R}}_{0,[n+1]}$ is connected for $r = 1, 9, 15, 21, 27, 33$ and it is not connected for $r = 25, 35$. In particular, all the above permit to see that there are exactly 32 values of $n \in \{4, \ldots, 100\}$ having not connected real locus, these values being given by $10, 14, 18, 22, 26, 28, 30, 34, 38, 42, 44, 46, 50, 52, 54, 58, 62, 66, 68, 70, 74, 76, 78, 82, 84, 86, 88, 90, 92, 94, 98, 100$.

Remark 15 (On the field of moduli and fields of definition)**.** The group $\mathrm{Gal}(\mathbb{C})$, of field automorphisms of $\mathbb{C}$, acts naturally on Ω_n by the following rule: if $\nu \in \mathrm{Gal}(\mathbb{C})$ and $\lambda = (\lambda_1, \ldots, \lambda_{n-2}) \in \Omega_n$, then $\nu(\lambda_1, \ldots, \lambda_{n-2}) := (\nu(\lambda_1), \ldots, \nu(\lambda_{n-2}))$. The field of moduli $\mathcal{M}_\lambda$ of the point $\lambda \in \Omega_n$ is the fixed field of the group $\{\nu \in \mathrm{Gal}(\mathbb{C}) : \nu(\lambda) = T(\lambda);\ \text{some } T \in \mathbb{G}_n\}$. A field of definition of λ is any subfield $\mathbb{K}$ of $\mathbb{C}$ such that there is an irreducible non-singular projective algebraic curve X of genus zero defined over $\mathbb{K}$ and there is an isomorphism $\psi : \widehat{\mathbb{C}} \to X$ such that the set $\{\psi(\infty), \psi(0), \psi(1), \psi(\lambda_1), \ldots, \psi(\lambda_{n-2})\}$ is invariant under the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{K})$ (the subgroup of all those field automorphisms of $\mathbb{C}$ acting as the identity on $\mathbb{K}$). It can be seen that $\mathcal{M}_\lambda$ is contained inside every field of definition of it and that it is the intersection of all its fields of definition [**37**]. In particular, $\mathcal{M}_\lambda \leq \mathbb{R}$ if and only λ is the fixed point of an anti-holomorphic automorphism of Ω_n, and $\mathbb{R}$ is a field of definition of λ if and only if $\{\infty, 0, 1, \lambda_1, \ldots, \lambda_{n-2}\}$ is kept invariant under an anticonformal involution of the Riemann sphere, that is, if and only if λ is a fixed point of a symmetry of Ω_n. Then, as observed in Remark 7, if is $n \geq 4$ is even, then $\mathcal{M}_\lambda \leq \mathbb{R}$ implies that $\mathbb{R}$ is a field of definition for $\lambda \in \Omega_n$.

CHAPTER IV

Applications

IV.1. Application 1: Generalized Fermat curves of type (k, n)

Definition 21 (Generalized Fermat curves of type (k, n))**.** *A closed Riemann surface S is called a generalized Fermat curve of type (k, n), where $k, n \geq 2$ are integers, if it admits a group $H \cong C_k^n$ of holomorphic automorphisms (see I.2.3.2) such that the quotient orbifold S/H has genus zero (then you can identify with the Riemann sphere $\widehat{\mathbb{C}}$) and it has exactly $n+1$ cone points, each one necessarily of order k; we say that H is a generalized Fermat group of type (k, n). If $(k-1)(n-1) > 2$, then in [29] it was observed that S is non-hyperelliptic and in [34] it was proved that S has a unique generalized Fermat group of type (k, n).*

As a consequence of the Riemann-Hurwitz formula [40, Theorem 4.16], the genus of a generalized Fermat curve of type (k, n) is:

$$g = g_{k,n} = 1 + \frac{k^{n-1}((k-1)(n-1)-2)}{2}. \tag{IV.1}$$

When $g_{k,n} > 1$, that is, $(k-1)(n-1) > 2$, it is said that Fermat's generalized curve of type (k, n) is **hyperbolic** (that is, its universal coverage is the hyperbolic plane $\mathbb{H}$, so it is standardized by a Fuchsian group).

An example of generalized Fermat curve of type $(k, 2)$ is given by the classic Fermat curve

$$X = \{[x; y; z] \in \mathbb{P}^2 : x^k + y^k + z^k = 0\} \subset \mathbb{P}^2,$$

which is hyperbolic when $k \geq 4$.

Remark 16. Generalized Fermat curves of type (k, n) of genus $g_{k,n} \leq 1$ correspond to the pairs $(k, n) \in \{(2,2), (2,3), (3,2)\}$. These cases will not be considered in this project since they are very well studied (the Riemann sphere and elliptic curves).

The generalized Fermat curves of the type (k, n) generalize in a certain sense to the classic curves of Fermat, and the concept of hyperelliptic surface.

IV.1.1. Algebraic description of generalized Fermat curves. Consider the fiber product of $(n-1)$ Fermat's classic curves, that is, an algebraic curve of the form:

$$\mathfrak{C}^k(\lambda_1, ..., \lambda_{n-2}) := \left\{ [x_1; ...; x_{n+1}] \in \mathbb{P}^n : \begin{array}{rl} x_1^k + x_2^k + x_3^k & = 0 \\ \lambda_1 x_1^k + x_2^k + x_4^k & = 0 \\ \lambda_2 x_1^k + x_2^k + x_5^k & = 0 \\ \vdots & \\ \lambda_{n-2} x_1^k + x_2^k + x_{n+1}^k & = 0 \end{array} \right\} \subset \mathbb{P}^n, \tag{IV.2}$$

where $k \geq 2$, and $\lambda_j \in \mathbb{C} \setminus \{0, 1\}$ such that $\lambda_j \neq \lambda_i$, for $i \neq j$. The conditions in the parameters λ_j ensure that $\mathfrak{C}^k(\lambda_1, ..., \lambda_{n-2})$ is a smooth algebraic curve, then a compact Riemann surface. In addition, this admits as a subgroup of holomorphic automorphisms to the group $H_0 \cong \mathbb{Z}_k^n$, which is generated by the linear transformations $a_1, ..., a_n \in GL_{n+1}(\mathbb{Q}(\omega_k))$, where $\omega_k = e^{2\pi i/k}$ and,

$$a_j([x_1; ...; x_{n+1}]) = \left[x_1; ...; x_{j-1}; \omega_k x_j; x_{j+1}; ...; x_{n+1}\right], \quad j = 1, ..., n.$$

If $a_{n+1}([x_1; ...; x_{n+1}]) = [x_1; ...; \omega_k x_{n+1}]$, then $a_1 a_2 ... a_{n+1} = 1$.

The fixed points of a_j, on $\mathfrak{C}^k(\lambda_1, ..., \lambda_{n-2})$, are given by:

$$Fix(a_j) = \left\{\left[x_1; ...; x_{j-1}; 0; x_{j+1}; ...; x_{n+1}\right] \in \mathfrak{C}^k(\lambda_1, ..., \lambda_{n-2})\right\}, \quad j = 1, ...n+1.$$

The application:

$$\widetilde{\pi} : \mathfrak{C}^k(\lambda_1, ..., \lambda_{n-2}) \to \widehat{\mathbb{C}} : [x_1; ...; x_{n+1}] \mapsto -\left(\frac{x_2}{x_1}\right)^k,$$

is a regular branched covering with deck group H_0 such that

$$\widetilde{\pi}(Fix(a_1)) = \infty, \quad \widetilde{\pi}(Fix(a_2)) = 0, \quad \widetilde{\pi}(Fix(a_3)) = 1,$$
$$\widetilde{\pi}(Fix(a_4)) = \lambda_1, \quad ..., \quad \widetilde{\pi}(Fix(a_{n+1})) = \lambda_{n-2},$$

i.e., the branched values of $\widetilde{\pi}$ are $\{\infty, 0, 1, \lambda_1, ..., \lambda_{n-2}\}$.

Remark 17. If a point in $\mathfrak{C}^k(\lambda_1, ..., \lambda_{n-2})$ is fixed by a non-trivial element of H_0, then it is fixed by some a_j.

Proposition 8. [29] *The Riemann surface defined for $\mathfrak{C}^k(\lambda_1, ..., \lambda_{n-2})$ is a Generalized Fermat curves of type (k, n) where H_0 is a Generalized group of Fermat of type (k, n). Let $Aut_{H_0}(\mathfrak{C}^k(\lambda_1, ..., \lambda_{n-2}))$ the normalizer of H_0 en* $\mathrm{Aut}^+(\mathfrak{C}^k(\lambda_1, ..., \lambda_{n-2}))$. *Then,*

(1) *$Aut_{H_0}(\mathfrak{C}^k(\lambda_1, ..., \lambda_{n-2}))/H_0$ is isomorphic to the subgroup of Möbius transformations that preserves the set $\{\infty, 0, 1, \lambda_1, \cdots, \lambda_{n-2}\}$; and*

(2) $Aut_{H_0}(\mathfrak{C}^k(\lambda_1, ..., \lambda_{n-2})) < PGL(n+1, \mathbb{C})$.

Like every pair of generalized Fermat (S, H) of type (k, n) is uniquely determined by the orbifold S/H, we obtain the following description of Fermat pairs of generalized type (k, n):

Theorem 8. [**29**] *Let S be a generalized Fermat curve of type (k, n), let H_S be a generalized Fermat group of type (k, n) for S and let $\pi : S \to \widehat{\mathbb{C}}$ be a regular branched cover with deck group H_S. If $p_1, \ldots, p_{n+1}$ are the branch values of π and M is the Möbius transformation so that $M(p_1) = \infty, M(p_2) = 0, M(p_3) = 1, M(p4) = \lambda_1, ..., M(p_{n+1}) = \lambda_{n-2}$, then S is isomorphic to $\mathfrak{C}^k(\lambda_1, ..., \lambda_{n-2})$. Moreover, such an isomorphism can be chosen to sent H_S to the generalized Fermat group H associated to the curve.*

Remark 18. The theorem 8 indicates that every Generalized pair of Fermat type (k, n) you can see, modulo isomorphisms, as a pair $(\mathfrak{C}^k(\lambda_1, ..., \lambda_{n-2}), H_0)$, for certain values $\lambda_1, \ldots, \lambda_{n-2}$.

IV.1.2. Hyperbolic generalized Fermat curves. Let S be a hyperbolic generalized Fermat curve of type (k, n). In [**34, 29**] it was observed that S has a unique generalized Fermat group H of type (k, n). Because of the uniqueness of the group H, we have that H is a normal subgroup of $Aut^+(S)$. In this way, we can consider the quotient group $Aut^+(S)/H$ (which is a finite group of transformations of Möbius), called the **reduced group of automorphisms** of S. As the finite subgroups of Möbius transformations are: the trivial group, the cyclic group $\mathbb{C}_n$, the dihedral group D_m (from order $2m$), the alternating groups $\mathcal{A}_4$ and $\mathcal{A}_5$ and the symmetric group $\mathfrak{S}_4$ (see, for example, [**6, 52**]), we have the structure of the reduced group of holomorphic automorphisms of a hyperbolic generalized Fermat curve of type (k, n).

As mentioned in IV.2, a compact Riemann surface S of genus $g \geq 2$ is called hyperelliptic if admit a unique holomorphic involution with exactly $2(g+1)$ fixed points, called the hyperelliptic involution. A hyperbolic generalized Fermat curve of type (k, n) is not hyperelliptic [**29**].

IV.1.2.1. *Fuchsian representation of hyperbolic generalized Fermat curves.* Given a hyperbolic generalized Fermat curve of type (k, n), we proceed to describe a description of the pair in terms of Fuchsian groups.

A **Fuchsian group** Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ (group of holomorphic automorphisms of $\mathbb{H}$), and it is said that Γ is **co-compact** if the quotient $\mathbb{H}/\Gamma$ is a compact surface.

Proposition 9. [36] *Let* Γ *a subgroup of* $\mathrm{PSL}_2(\mathbb{R})$. *Then*

(1) Γ *is a Fuchsian group if and only if act properly discontinuously on* $\mathbb{H}$.
(2) If Γ *is a Fuchsian group, then the quotient* $\mathbb{H}/\Gamma$ *is a orbifold and has the Riemann surface structure for which the canonical projection* $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$ *is a branched covering.*
(3) A Fuchsian group acts freely on $\mathbb{H}$ *if and only if* Γ *it is free of torsion (that is, it does not have nontrivial elements of finite order). In this case,* $\mathbb{H}/\Gamma$ *has no conical points, that is, it is a Riemann surface (whose fundamental group is isomorphic to* Γ*).*

Let (S, H) be a pair of hyperbolic generalized Fermat of type (k, n). As a consequence of the Uniformization Theorem of Klein-Koebe-Poincaré, there is a Fuchsian group co-compact free of torsion $L \leq \mathrm{PSL}_2(\mathbb{R})$ such that $S \cong \mathbb{H}/L$.

If $N(L)$ is the normalizer of L on $\mathrm{PSL}_2(\mathbb{R})$ (which is a Fuchsian group), then there is a surjective homomorphism

$$\theta : N(L) \to Aut^+(S)$$

con $Ker(\theta) = L$. So, $N(L)/L \cong Aut^+(S)$.

If $\Gamma = \theta^{-1}(H)$, then $L \leq \Gamma \leq N(L)$ (and Γ is a Fuchsian group) and

$$\theta\,|_{\Gamma}: \Gamma \to H$$

is a surjective homomorphism with kernel L. So, $\Gamma/L \cong H$ y $S/H \cong \mathbb{H}/\Gamma$. Also, as the orbifold quotient S/H has a signature $(0; k, \overset{n+1}{\ldots}, k)$, then the group Γ have a presentation of the form:

$$\Gamma = \langle x_1, ..., x_{n+1}; x_1^k = ... = x_{n+1}^k = x_1 x_2 ... x_{n+1} = 1 \rangle. \tag{IV.3}$$

If Γ' is the subgroup derived from Γ, then $\Gamma/\Gamma' \cong \mathbb{Z}_k^n \cong H \cong \Gamma/L$. So, $\Gamma' = L$ and $H \cong \Gamma/\Gamma'$, that is, $(S, H) \cong (\mathbb{H}/\Gamma', \Gamma/\Gamma')$ [29].

The above, together with the uniqueness of the generalized Fermat group , allows us to obtain the following fact.

Theorem 9. [29]*[Isomorphism of generalized Fermat curves] Let* $k, n \geq 2$ *be integers such that* $(k-1)(n-1) > 2$, S_j *a generalized Fermat curve of type* (k, n)*, with* $H_j \cong \mathbb{Z}_k^n$ *its only generalized Fermat group of type* (k, n) *y* $\pi_j : \mathrm{S}_j \to \widehat{\mathbb{C}}$ *a regular branched covering with deck group* H_j*, for* $j = 1, 2$*. Then* S_1 *and* S_2 *they are isomorphic if and only if there is a Möbius transformation that sends the branching values of* π_1 *in the branching values of* π_2*.*

IV.1.2.2. *Moduli Space of hyperbolic generalized Fermat curves.* Of Theorem 8 and of Theorem 9, we observe the following:

Let S be a hyperbolic generalized Fermat curve of type (k,n) and $H < \mathrm{Aut}^+(S)$ its generalized Fermat group of type (k,n). Let $\pi : S \to \widehat{\mathbb{C}}$ be a regular branched covering with deck group H and branching values $p_1, ..., p_{n+1}$ and $M \in \mathrm{PSL}_2(\mathbb{C})$ a Möbius transformation such that $M(p_1) = \infty, M(p_2) = 0, M(p_3) = 1, M(p_4) = \lambda_1, ..., M(p_{n+1}) = \lambda_{n-2}$. Then $(\lambda_1, ..., \lambda_{n-2}) \in \Omega_n$ and there is an isomorphism $\phi : S \to \mathfrak{C}^k(\lambda_1, ..., \lambda_{n-2})$ that conjugates H on H_0.

As in the hyperbolic case, the generalized Fermat group is unique, and in the case of genus one all its groups of Fermat generalized with conjugates by a holomorphic automorphism, the above gives us the following.

Corollary 4. [**29**] *If $k \geq 2$ and $n \geq 3$ are integers, then Ω_n provides a parameter space for generalized Fermat curves of type (k,n).*

As a result of the previous results we have the following fact.

Theorem 10. [**29**] *Let $k \geq 2$ and $n \geq 3$ be integers. Then two points of Ω_n define generalized Fermat curves of type (k,n) isomorphic if and only if they are in the same $\mathbb{G}_n$-orbit. In particular, the Moduli space of the generalized Fermat curves of type (k,n), $\mathcal{F}_{k,n}$, is isomorphic to the quotient space $\Omega_n/\mathbb{G}_n$. And the branch locus $\mathcal{B}_{0,[n+1]}$ consists of those admitting more conformal automorphisms than the generalized Fermat group of type (k,n).*

Theorem 11 (Corollary of the theorems 6 and 7)**.** *The locus in $\mathcal{F}_{k,n}$, consisting of those admitting more conformal automorphisms than generalized Fermat group of type (k,n), is connected for $n \geq 4$ even and for $n \geq 6$ divisible by 3, and it has exactly two connected components otherwise. Its real locus is connected for $n \geq 5$ odd, and it is not connected for $n = 2r$, $r \geq 5$ odd.*

Remark 19. This results can be applied as well to the more unknown (and difficult to work with) generic p-gonal curves, simple generic p-gonal curves [**17**, **18**, **39**, **21**].

IV.2. Application 2: Hyperelliptic Riemann surfaces

Definition 22 (Hyperelliptic Riemann surfaces)**.** *A Riemann surface compact of genus* $g \geq 2$ *is called* **hyperelliptic** *if it admit a single holomorphic involution with exactly* $2(g+1)$ *fixed points, called the* **hyperelliptic involution**. *Equivalently,* S *is hyperelliptic if and only if there is a regular branched covering of two sheets* $\pi : S \to \widehat{\mathbb{C}}$ *with* $2(g+1)$ *branched values.*

Remark 20. Let S be a Hyperelliptic Riemann surfaces of genus $g \geq 2$, let a regular branched covering of two sheets $\pi : S \to \widehat{\mathbb{C}}$ with $2(g+1)$ branched values, let's say $\{p_1, ..., p_{2(g+1)}\}$. The branched values can be normalized, this is, there is a Möbius transformation that send the branched values to the set of points $\{\infty, 0, 1, \lambda_1, ..., \lambda_{2g-1}\}$. Every time we have a holomorphic automorphism of S induces a holomorphic automorphism of $\widehat{\mathbb{C}}$ (a Möbius transformation) keeping invariant the $2(g+1)$ points (branched values), and on the contrary also, that is, every time we have a Möbius transformation keeping invariant the $2(g+1)$ points then comes from a holomorphic automorphism of S.

IV.2.1. Algebraic model of the hyperelliptic Riemann surfaces. The algebraic equation of a hyperelliptic Riemann surface S is of the form (if all $p_j \in \mathbb{C}$)

$$y^2 = \prod_{j=1}^{2(g+1)} (x - p_j),$$

where p_j are the branched values of the regular branched covering of two sheets $\pi : S \to \widehat{\mathbb{C}}$.

IV.2.2. Moduli space of the hyperelliptic Riemann surfaces. Let S_1 and S_2 be hyperelliptic Riemann surfaces of genus $g \geq 2$, with its regular branched coverings of two sheets $\pi_1 : S_1 \to \widehat{\mathbb{C}}$ with the branched values the set of points $\{\infty, 0, 1, \lambda_1, ..., \lambda_{2g-1}\}$ and $\pi_2 : S_2 \to \widehat{\mathbb{C}}$ with the branched values the set of points $\{\infty, 0, 1, \mu_1, ..., \mu_{2g-1}\}$ respectively, then, if there is an isomorphism f between S_1 and S_2 then it will conjugate the hyperelliptic involution of S_1 to the hyperelliptic involution of S_2 with exactly $2(g+1)$ fixed points, then f descends to a Möbius transformation that sends the set $\{\infty, 0, 1, \lambda_1, ..., \lambda_{2g-1}\}$ in $\{\infty, 0, 1, \mu_1, ..., \mu_{2g-1}\}$. And the opposite also happens, that is, if we have a Möbius transformation that sends the set $\{\infty, 0, 1, \lambda_1, ..., \lambda_{2g-1}\}$ in the set $\{\infty, 0, 1, \mu_1, ..., \mu_{2g-1}\}$ then we can construct an isomorphism f of S_1 in S_2.

Theorem 12. [**12**, §2] *Two hyperelliptic Riemann surfaces* S_1 *and* S_2 *are isomorphic if and only if there is a Möbius transformation that sends the set of branched values of* $\pi_1 : S_1 \to \widehat{\mathbb{C}}$ *to set of the branched values of* $\pi_2 : S_2 \to \widehat{\mathbb{C}}$.

Remark 21. Let $n + 1 = 2(g+1)$, with $g \geq 2$, then the following holds.

(1) For every $(\lambda_1, ..., \lambda_{2g-1}) \in \Omega_n$ we may build the set $\{\infty, 0, 1, \lambda_1, ..., \lambda_{2g-1}\}$ and obtain a hyperelliptic Riemann surfaces S. Moreover, each hyperelliptic Riemann surface of genus g is isomorphic to one as above. So Ω_n parametrizes the set of hyperelliptic Riemann surfaces.
(2) As a consequence of the above theorem, the quotient $\Omega_n/\mathbb{G}_n$ provides a model for the moduli space $\mathcal{H}_g$ of the hyperelliptic Riemann surfaces of genus g.

The above asserts that if $n = 2g+1$, where $g \geq 2$, then $\mathcal{M}_{0,[n+1]}$ can be identified with the moduli space $\mathcal{H}_g$ of hyperelliptic Riemann surfaces of genus g. The branch locus $\mathcal{B}_{0,[n+1]}$ consists of those hyperelliptic Riemann surfaces admitting more conformal automorphisms than the hyperelliptic one. The description of the groups of conformal automorphisms of hyperelliptic Riemann surfaces can be found in [**10**]. Theorems 6 and 7 assert the following simple fact.

Theorem 13 (Corollary of the theorems 6 and 7)**.** *The locus in $\mathcal{H}_g$, consisting of those hyperelliptic Riemann surfaces admitting more conformal automorphisms than the hyperelliptic one, is connected if $2g + 1$ is divisible by 3 and it has exactly two connected components otherwise. The real locus in $\mathcal{H}_g$ is connected.*

The above result is related to the ones obtained in [**19**] by Costa, Izquierdo and Porto, where they prove that the hyperelliptic branch locus of orientable Klein surfaces of algebraic genus $g \geq 2$ with one boundary component is connected (in the case of non-orientable Klein surfaces they proved that it has $(g+1)/2$ components, if g is odd, and $(g+2)/2$ components otherwise).

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